Anabelian Geometry
A modern overview

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Caveat emptor. *Dense and not self-explanatory slides ahead; additional examples and comments were written during the talk.*

Jump here

What is arithmetic anabelian geometry

For X algebraic variety over k: *does (parts of) the Arithmetic-geometric Ex. Seq.*:

$$1 \to \pi_1^{et}(X \otimes \bar{k}, *) \to \pi_1^{et}(X, *) \stackrel{pr}{\to} Gal(\bar{k}/k) \to 1 \qquad \text{(AG)}$$

reconstructs (parts of) the variety X? [relative/semi-abs./absolute]

Some insights.

- *Tate Conjecture:* A, B abelian varieties. `Faltings'83`

$$Hom_K(A,B) \otimes \widehat{\mathbb{Z}} \simeq Hom_{G_k}[H_1(A_{\bar{K}}, \widehat{\mathbb{Z}}), H_1(B_{\bar{K}}, \widehat{\mathbb{Z}})]$$

- *Analytic/Grp. Th. input:* For X smooth curve of genus g. `SGA1`
  Riemann existence theorem $\rightsquigarrow \Delta_X \simeq \widehat{\pi_1}^{top}(X(\mathbb{C})^{an})$
  The r-cusps inertia gives a system of generators of Δ_X.

$\Rightarrow$ **Typical context:** Take X *hyperbolic* curve (2g-2+r>0), i.e Δ_X is not abelian.

In today's talk:

1. *Anabelian principles* `Nakamura'80s;Tamagawa,Mochizuki'90s`
   $\rightsquigarrow$ "There exists a functorial group theoretic algorithm"
2. *A new anabelian Diophantine geometry* `Mochizuki,Hoshi 2010-20`
   Inter-universal Teichmüller geometry, beyond ring structures
3. *Anabelian, $\mathbb{A}^1$ and motivic geometries...* (...)

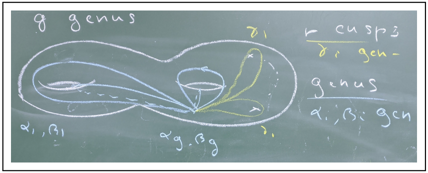
Grothendieck-Teichmüller th. `Hoshi,Minamide,Mochizuki 17+;-Tsujimura 21+`
(AG) for $X \in \mathcal{M} = \{\mathcal{M}_{g,m}\}_{g,m}$ moduli spaces of curves: $G_{\mathbb{Q}} \leqslant \widehat{GT} \leqslant Out[\pi_1(\mathcal{M})]$

Bi-anabelian geometry over global fields

Let X_i be *affine* (r>0) hyperbolic curves over K f.g. field over $\mathbb{Q}$, then:

$$Isom_K(X_1, X_2) \xrightarrow{\sim} Isom_{G_K}[\Pi_{X_1}, \Pi_{X_2}]_\sim \qquad (Bi_K)$$

The (g,r)-geometrical context



Smooth comp. $X \hookrightarrow \bar{X}$:

- $g = genus(\bar{X})$,
- cusps $r = \#(\bar{X} \setminus X)(\bar{K})$

For $x \in \bar{X}$, in Δ_X and Π_X:

- decomp. $D_x = Stab_{\Pi_X}(x)$
- inertia $I_x = D_x \cap \Delta_X$ groups.

Reconstruction I [Nakamura]

Grp th. & weights

1. $\Pi_X \rightsquigarrow \Delta_X < \Pi_X$. As max. top. f.g. $H \triangleleft^{cl} \Pi_X$
2. $\Pi_X \rightsquigarrow (g, r)$. Abelianization and Deligne Frob. weights in étale homology:

$$0 \to \mathbb{Q}_\ell(1) \to \oplus_x \mathbb{Q}_\ell(1) \to \Delta_X^{ab} \otimes \mathbb{Q}_\ell \to T_\ell(Jac_\Delta) \otimes \mathbb{Q}_\ell \to 0$$

   Distinguishes genus and inertia generators $\rightsquigarrow 2g$ and $2g - r + 1$.
3. $\Pi_X \rightsquigarrow I_x$. I is cuspidal if self-normalizing and $I \simeq \widehat{\mathbb{Z}}(1)$.

For g(X)=0: Steps 1.-3. & a $(K^\times, \boxtimes)$ argument recover $X \simeq \mathbb{P}^1 \setminus \{0, 1, \infty, \underline{\lambda}\}$

Reconstruction II [Tamagawa] *Good red. & Rat. pts & Func. field*

1. *Oda's anabelian good reduction.* For v place of K, a "Serre-Tate" Out-test.
   (Bi_v) $\forall' v \in \mathbb{V}(K)$: $\pi_1^{(tame)}(X_{k_v})$ the red. over k_v **finite field** $\rightsquigarrow X_{k_v} \Rightarrow (Bi_K)$
2. *Section and limits of k-points.* For $s\colon G_k \to \Pi_X$ a section:
   $^\forall(Im\ s < H) \lhd^{op} \Pi_X : Y_H(k) \to X \neq \emptyset$ by *Lefschetz* tr. on ℓ-cohom.
   $\rightsquigarrow x_\infty = \lim_{Im\ s<H} y_H(k) \in X(k) \rightsquigarrow$ *Decomposition* $D_x =< Im\ s >$.
3. *The* $(K(X_v)^\times, \boxtimes)$. Local class field theory as $K(X_v)^\times = Ker\ \Psi$

$$\prod \widehat{K}_x^\times / \widehat{O}_x^\times \times \prod \widehat{K}^\times \xrightarrow{\Psi} \Pi_X^{ab}$$

   Then *the field* $K(X_v)$ via (techn.) valuation and functions.

Reconstruction III [Mochizuki] *Containers & Line Bdles*

Let X_i be proper hyperbolic curves over K a sub-p-adic field.

1. *Hodge-Tate container for* X. $(\Delta_X^{(p),ab} \otimes \mathbb{C}_p)^{G_K} \simeq H^0(X, \omega_X) \rightsquigarrow X \overset{\exists}{\hookrightarrow} \mathbb{P}(\Gamma_X^\Omega)$
2. *A p-adic Tamagawa rat. construction.* A M-split $\rightsquigarrow x_\infty \in \mathbb{P}(\Gamma_X^\Omega)(L)$
3. *Chern cl. of LB.* A L-line bundle give a L-point (grp. th.)

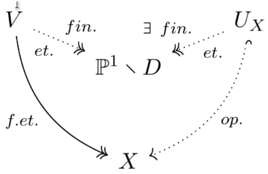
- Use "only" $\Delta_X/[\Delta_X, [\Delta_X, \Delta_X]] \rightsquigarrow$ "minimal reconstruction?" T-Saïdi,Y.'20
- Use a "quotient $\mathcal{Z}_X$" in the Malcev of $\Delta_X \rightsquigarrow$ "Lie anab. tech.?" Sawada'20.

Mono-anabelian Geometry. Reconstruction of X starting from Π_X only.
A. Tamagawa: "should be considered as involving all objects: One for all, all for one!

From Π_X to function & base fields $K(X)$ and k. For X hyperbolic over k MLF.

- *Inertia groups I_x.* Belyi cuspidalizations $\{\Pi_{U_X} \to \Pi_X\}_{U_X \hookrightarrow X} \rightsquigarrow$ set of I_x, $x \in X \setminus U_X$.
- *Multiplicative groups $K(X)^\times$ & $k^\times$.* A container:
  $\Gamma(U, \mathcal{O}_U) \overset{\kappa}{\hookrightarrow} H^1(\Pi_U, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))$
  $k^\times \hookrightarrow H^1(G_k, \mu_{\widehat{\mathbb{Z}}}(\Pi_X)) \hookrightarrow H^1(\Pi_X, \mu_{\widehat{\mathbb{Z}}}(\Pi_X))$
  ...After *cycl. sync.* $\mu_{\widehat{\mathbb{Z}}}(\Pi_X) \simeq I_x$ for $x \in \bar{X} \setminus X$.
- *The fields $K(X)$ and k.* Uchida (val. tech.)

Curve of strict Belyi type.



k MLF - From G_k to ... For $G_k \curvearrowright \mathcal{O}_k^\times < \mathcal{O}_k^\triangleright < \mathcal{O}_k < k^\times$ as ⊠-monoids:
Local Class Field th.$\Rightarrow$ $G \rightsquigarrow G \curvearrowright \mathcal{O}^\times(G)$, $\mathcal{O}^\triangleright(G)$, $k^\times(G)$ as G-monoids

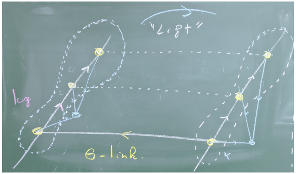
1. But Γ-orbits of Étale/Frobenius isom. of cyclotomes $\Lambda(G) \simeq \Lambda(M)$ for $G \curvearrowright M$
2. But no (Ring, ⊞, ⊠) nor field reconstructions! *(bi-anab. for MLF)*

Category of Quasi-tripods. Hoshi'19+
Curves X st $X \rightsquigarrow \exists X_1 \rightsquigarrow \dots \rightsquigarrow X_n = \mathbb{P}^1_{\{0,1,\infty\}}$ with $\rightsquigarrow \in \{f.et, (f.et)^{-1}, op.im.\}$
$\Rightarrow$ *"Any smooth variety has an absolute anabelian basis"* *(Higher dim. bi-anab.)*

Mono-anabelian results lead to a Diophantine *heights inequality*:

$$\mathrm{ht}_{\omega_X(D)} \lesssim (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)$$

The log-theta lattice of HTs.



Log-theta latt. of Hodge-theta theatres $\{{}^{\circ,\circ}\mathcal{HT}\}$:

Cat. Gl. of $\begin{cases} G_{k_v} \curvearrowright \mathcal{O}^{\triangleright}_{k_v} \\ \Pi_v = \pi_1^{tp}(X_{k_v}) \end{cases}$ for $(1,1)$-curves X/K, v a K-place.

Nb. $^{\exists}$*containers* & cycl. rigidity for *theta*-fct. of $E(k_v)^{rig} \simeq \mathbb{G}_m/q^{\mathbb{Z}}$ wrt π_1^{tp} with Π_X-comp.

Mono-anabelian group theoretic reconstruction $\rightsquigarrow$ **A multi-radial algorithm**

- *Mono-anab. transport* between Étale & Frobenius objects:

$$\mathcal{O}^{\times}_{\dagger k} \xrightarrow{\kappa} H^1(G_{\dagger k}, \Lambda({}^{\dagger}\mathcal{O})) \xrightarrow[\sim]{\widehat{\mathbb{Z}}^{\times}-Ind} H^1(G_{\ddagger k}, \Lambda({}^{\ddagger}\mathcal{O})) \xrightarrow{\kappa^{-1}} \mathcal{O}^{\times}_{\ddagger k}$$

- *For* $\boxtimes/\boxplus$*-monoids* (Kum., theta): $(G_v, \Pi_v, \Delta_v, \ldots) \rightsquigarrow (\mathcal{O}^{\triangleright}(G), \Delta_{\Theta}, \mathbb{M}(\Pi), \ldots)$
- *Log-shell container:* That contains: $(\mathcal{O}^{\triangleright}_{\dagger k}, \boxtimes) \stackrel{log}{\to} (\mathcal{O}^{\triangleright}_{\ddagger k}, \boxplus)$

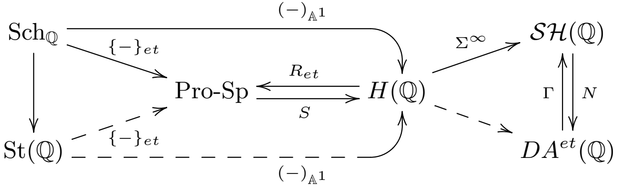
Results

1. **abc-conjecture** via a Galois-Vojta reformulation. Mochizuki'12+
2. **Fermat Last Theorem** via IUT-refined (Belyi & theta eval.) FHMP'21+
3. **Grothendieck Section Conj (Loc.-Glob. birat.)** via IUT-Galois. Hoshi-M

For X non $K(\pi,1)$ and of higher dimension, replace π_1^{et} by the *étale topological type* $\{X\}_{et} \in$ Pro-Sp $\rightsquigarrow$ Gives access to $\pi_N^{et}(X)$ for $N \geqq 1$.

Unifying the three insights of Grothendieck: Arithmetic geometry, motivic theory and Quillen model categories:

Friedlander's pro-spaces, Morel-Voevodsky's motivic homotopy, and Ayoub's motives



Spaces | Anab. Geo. | $\mathbb{A}^1$-Geo. | Motives

A few (prospective) results:

- *Higher dim.:* Anabelian "polycurves" over *number fields* `Schmidt-Stix'16`
- *Higher dim.:* non $\mathbb{A}^1$-rigid *anabelian obstruction* . . .
- *Higher sym.:* formalism *extends from schemes to stacks* . . .

... Requires further geometric foundational techniques ...

For Further Reading...

This list of references should be used as entry points for the **primary sources** ([pGC],...)

On anabelian geometry

- Y. Hoshi, Introduction to Mono-anabelian geometry, *Proc. "Fundamental Groups in Arithmetic Geometry"*, 2016.
- S. Mochizuki, Topics in absolute anabelian geometry I, II & III, 2012, 2013, 2015.
- H. Nakamura, A. Tamagawa and S. Mochizuki, The Grothendieck conjecture on the fundamental groups of algebraic curves, *Sugaku Expositions (AMS), 14* (2001).
  Survey with bibliographical references and notes.

On Inter-universal Teichmüller geometry

- Invitation to inter-universal Teichmüller Theory & Inter-universal Teichmüller Theory Summit, Collas, Fesenko, Hoshi, Mochizuki, Taguchi (org.), *RIMS Expanding Horizons of IUT workshop, (2021).* #IUT-WS1 - #IUT-WS2
  Notes and recordings of the talks of the 2 workshops (avail. on registration).
- B. Collas, Promenade in Inter-universal Teichmüller theory, RIMS-Lille seminar, 2021.
  Also provides an overview of absolute mono-anabelian reconstructions. *#Text*

Recent developments in Anabelian Arithmetic Geometry

- Homotopic and Geometric Galois Theory, Collas, Dèbes, Nakamura & Stix (edt.), *Oberwolfach report No. 12/2021*, (2021). #Text
  With references to Hoshi, Minamide, Tsujimura, Sawada's recent works.

Thank you for your attention :: ご清聴ありがとうございました