Mathematisches Forschungsinstitut Oberwolfach

MFO-RIMS Tandem workshop
Arithmetic Homotopy and Galois Theory

Organized by
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[List of reports of the workshop]

Abstract. This report presents a general panorama of recent progress in the arithmetic-geometry theory of Galois and homotopy groups and its ramifications. While still relying on Grothendieck's original pillars[1], the present program has now evolved beyond the classical group-theoretic legacy *to result in an autonomous project that exploits a new geometrization of the original insight and sketches new frontiers between homotopy geometry, homology geometry, and diophantine geometry.*

This panorama "closes the loop" by including the last twenty-year progress of the Japanese arithmetic-geometry school via Ihara's program and Nakamura-Tamagawa-Mochizuki's *anabelian approach*, which brings its expertise in terms of algorithmic, combinatoric, and absolute reconstructions. These methods supplement and interact with those from the classical *arithmetic of covers and Hurwitz spaces* and *the motivic and geometric Galois representations*.

This workshop has brought together the next generation of arithmetic homotopic Galois geometers, who, with the support of senior experts, are developing new techniques and principles for the exploration of the next research frontiers.

Mathematics Subject Classification (2020): 12F12, 14G32, 14H30, 14H45 (Primary); 14F05, 55Pxx, 14F22, 14G05, 14D15, 11F70 (Secondary).

[1]That are, as presented in "Récoltes et semailles", the resolution of the discrete and the continuous, the local-to-global thinking by generization-specialization, and the quintessential intersection of arithmetic and geometry – see § 2.10 ibid.

Introduction by the Organizers

The absolute Galois group $G_{\mathbb{Q}}$ of rational numbers is the seed of number theory. Its study by homotopic and geometric means is at the heart of modern arithmetic geometry. Building on the result of previous investigations, we report on recent progress on the development of a new geometry of Galois theory and homotopy symmetries of spaces. Relying on the unexploited work of Grothendieck's descendants, this results in *a new "geometrization" of arithmetic geometry based on the homotopy insight*[2].

This workshop, together with three previous ones[3], has led to the following cross-bridging principles:

(1) The application of *classical approaches beyond their original geometric frontier* (e.g. patching, Hilbert realization, Hurwitz spaces, section conjecture);
(2) Some *enriched approaches to linear Galois representations* (e.g. local systems via analytic automorphic forms, via the derived category of perverse sheaves, and via Tannaka symmetries);
(3) The *research of an intermediate new type of arithmetic geometry* (e.g. in terms of abelian-by-central extensions, of simplicial geometry in between étale and motivic theories, of the homotopy-homology frontier);
(4) The *absolute* reconstruction of *anabelian arithmetic-geometry* in a context that goes *beyond the ring structure* (e.g. applications in Diophantine geometry, anabelian combinatorial understanding of $\widehat{GT}$ and $G_{\mathbb{Q}}$.)

In the present report, these principles are supported further by *the integration of such techniques* as p-adic Hodge theory (for (2) and (3)), the monodromy method for dynamic arithmetic (for (1)), the development of ℓ-adic Galois theory and anabelian representations (for (3) and (4)), the development of anabelian reconstruction over algebraically closed field of positive characteristic (for (3)), and the introduction of indeterminacies for the uncoupling of multiplicative and additive structures in anabelian geometry and K-theory (for (1), (3) and (4)).

Joint together, these principles and techniques further indicate the following *new research frontiers in arithmetic geometry*:

(a) *Arithmetic & rationalization:* Hurwitz-Hilbert geometry, realization-lifting-parametrization proram for covers, and rational obstruction;
(b) *Homology & homotopy:* Galois and p-adic Hodge representations, monodromy and Tannaka symmetries, the meta-abelian frontier;

[2] Under this form this trend takes the denomination of "Arithmetic and homotopic Galois theory" (AHGT).

[3] MFO workshop - *Homotopic and Geometric Galois Theory* (org.: B. Collas, P. Dèbes, H. Nakamura, J. Stix) in 2021; "Rencontres Arithmétiques de Caen" on *Field Arithmetic and Arithmetic Geometry* in 2019; MFO mini-workshop *Arithmetic Geometry and Symmetries around Galois and Fundamental Groups* (org.: B. Collas, P. Dèbes, M. Fried) in 2018.

(c) *New geometries:* in higher dimension and with respect to stack symmetries, over algebraically closed and finite fields, for multiplicative and additive monoids (with indeterminacies).

For a better measure of the progress of this workshop, we keep the three classical and historical approaches, that are *the arithmetic of Galois covers and Hurwitz spaces*, the *geometric and motivic Galois representations* and the *anabelian arithmetic geometry* as a structuring guide. Given the constant mutual interactions of these topics, this categorization must be read up to certain mild indeterminacies; we refer to the original reports for details.

Arithmetic of Galois covers and Hurwitz spaces. Geometry of Hurwitz spaces and concatenation within their boundary are exploited for building new rational irreducible components (Seguin). A refined version of the ring of components and splitting number are developed, which has already ramified in algebraic topology and for enumerative questions in number theory – see the work of Bianchi and Ellenberg-Venkatesh-Westerland respectively. The latter, with the systemic use of homological stability, has since established in a well-identified program of "arithmetic statistic of function fields", with, among others, applications to Malle's conjecture and the distribution of Selmer group (Westerland et al.)

Originally motivated by question from arithmetic dynamic, structural results are obtained for the monodromy group of iterated polynomials in terms of Hilbert irreducibility theorem, largeness, and arboreal representations (König, Neftin et al.). In a similar but distinct probabilistic direction, we also refer to Bary-Soroker.

The development of two in-progress projects around properties of torsion points of the jacobian of curves have been reported: one on the extension of previous results of Greenberg which, in the spirit of Ihara's program, tightly intertwines number theory and arithmetic geometry (Pries et al.), and a second, by building on previous insight of Raynaud, Tamagawa and Hoshi, on possible strategies for solving the Coleman conjecture (Takao).

Geometric and motivic Galois representations. Concerning the "homology vs homotopy" frontier, this workshop has seen *the path between linear and anabelian methods* – a path initiated in the 90's on one side with the integration of Falting's p-adic Hodge method in anabelian geometry by Mochizuki and one the other side by Kim's non-anabelian approach of Chabauty-Coleman for rational points – *to be pushed closer to a loop*[4]. At the intersection of Lawrence-Venkatesh method for the proof of the Mordell conjecture and of Chabauty-Kim theory, progress was reported on the rational obstruction to Selmer sections that exploits the whole arsenal of p-adic Hodge theory (Betts et al.). A synthetic panorama of properties of the degeneracy/toric locus of ℓ-adic local systems that build on similar approaches based on variational p-adic Hodge theory and period map was also presented

[4]This topic will be the object of a dedicated AHGT workshop in Oberwolfach, "Anabelian Geometry and Representations of Fundamental Groups" Sep. 29 - Oct. 4, 2024 (Org.: A. Cadoret, F. Pop, J. Stix, A. Topaz; ID: 2440).

(Cadoret et al.). Exploiting a special type of geometric Galois representations (of Barsotti-Tate type), report was presented on *how arithmetic invariants resulting from explicit and computational approach* lead to new geometric and arithmetic results in the p-adic Langlands program (Mézard et al.)

On the classical topic of Artin L-functions, and as motivated by potential applications of anabelian geometry to analytic number theory, some investigations were presented that reflect the use of purely group-theoretic method (Yamashita).

In the direction of étale cohomology and the structure of Massey products, and with *motivation from the embedding problem*, an extensive and solid state-of-the-art of the most recent results was presented for Galois cohomology in terms of formality and with application to Koszulity (Quick et al.) and for cohomology of curves in various geometric situations (Bleher et al.). A synthetic report was given on finitness results of Galois cohomology and Tate-Shafarevich groups, a key tool whose properties ramifies in multiple aspects of the AHGT program (Harari et al.)

Anabelian arithmetic geometry & ramifications. This workshop has been the opportunity to report on *two recent breakthroughs*: the construction, as a consequence of the anabelianity of the Grothendieck-Teichmüller group, of *a combinatorial model of* $\bar{\mathbb{Q}}$ (Tsujimura et al.), a decisive step toward the Galois-Grothendieck-Teichmüller conjecture, and *the resolution of non-singularities* (Lepage), an algebraic geometry statement originally formulated by Tamagawa, with implications in anabelian geometry, Grothendieck-Teichmüller theory, and the section conjecture.

A certain thematic group has appeared, that exploits various flavors of anabelian ℓ-adic Galois representations, in terms of purely anabelian methods (Iijima et al.), of the symmetries of spaces (applied to associators, Shiraishi or to Oda's conjecture, Philip et al.), or in relation with Deligne-Ihara's conjecture pushed from genus zero to one (Ishii).

The emergence of a new geometry of monoids with indeterminacies provides new connections between anabelian geometry, diophantine geometry and analytic number theory (Mochizuki et al.), a principle that can also be found in the reconstruction of function fields via K-theory (Topaz).

New geometric frontiers were exploited: higher homotopy and motivic rational obstructions via the simplicial homotopy method (Corwin), the nearly-abelian study of local-global properties of Galois sections (Porowski), and a reconstrution program over algebraically closed fields of positive characteristic, with in particular a new proof of Mochizuki's seminal anabelian result (Yang et al.).

A bridge between Europe and Japan. Following the Oberwolfach tradition, the workshop was opened with a few words of Prof. Klaus on behalf of MFO and of Prof. K. Ono on behalf of RIMS, both present at RIMS Kyoto. The workshop was structured over 2 sites, one in Japan and one in Germany, with a Zoom bridge for live interaction and a dedicated Discord forum for sharing video recordings of the talks, slides, and asynchronous comments. The crossover of some Japanese researchers at the MFO and of some French researchers in Japan ensured the dissemination of ideas between the two sites.

The week gathered a total of 58 participants – 25 participants at RIMS Kyoto and 33 participants at Oberwolfach Germany – around 25 one-hour long talks (5 each day). An extended break time after lunch – under the form of a "Bento" time with MFO-like random seating at RIMS – supported informal interactions and discussions during the event.

Speakers, rather than to restrict themselves to their individual work, reported on recent progress of entire subtopics. Scientific exchanges reached the next stage, where participants would send video recordings of their comments and questions on blackboard to the other site.

Poster session for Oberwolfach Leibniz Fellows. The oral presentations above were complemented with online poster sessions on the dedicated MFO-RIMS Discord forum for the OWLG fellows to introduce their research topics and latest results: (1) Assoun (Lille) uses Galois theory of skew fields for the inverse Galois problem, (2) Holzschuh (Heidelberg) develops the étale homotopy type of spaces in terms of infinity categories for a higher-dimensional result on Grothendieck's section conjecture, and (3) Shmueli (Tel Aviv) obtains probabilistic results on the residue degree and ramification of p-adic splitting field of polynomials.

A decisive opus within the AHGT project. This third opus has confirmed the dynamic initiated with the 2018 mini-workshop "Arithmetic Geometry and Symmetries around Galois and Fundamental Groups" and developed in the 2021 workshop "Homotopic and Geometric Galois Theory". A shared feeling of the participants on both sites is that *a new common culture has been built, a structured program has appeared that paves the way to future collaboration and a network of conjectures.* Following the strong support and feedback of the participants, and as part of the "Arithmetic and Homotopic Galois Theory" project (AHGT)[5], agreement has been made to meet again within the next 2 years for reporting on the latest progress of the field.

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[5]For updated information on workshops, seminars, and publications of the AHGT project, we refer to https://ahgt.math.cnrs.fr