SYMMETRIES OF SPACES AND NUMBERS – ANABELIAN GEOMETRY

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Abstract. *"Can number and geometric spaces be reconstructed from their symmetries?"* This question, which is at the heart of anabelian geometry, a theory envisioned by Alexander Grothendieck and developed in many variants by the Japanese arithmetic school, illustrates, in the case of a positive answer, the universality of the homotopic method in arithmetic geometry.

Starting with elementary examples, this paper first introduces the motivations and guiding principles of the theory, then presents its most structuring results and its contemporary trends.

As a result, the reader is presented with a rich and diverse landscape of mathematics, which thrives on theoretical and explicit methods, and runs from number theory to topology.

Contents

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1. Reconstructions from symmetries

At its most elementary level, for numbers and for spaces, anabelian geometry deals with properties of polynomials. We investigate how the geometric notion of symmetries can be applied to the case of numbers. The shadow of a unifying context begins to appear, which will be fully completed in the next section.

1.1. **From roots to symmetries - Galois theory.**
While it is known that the field of complex numbers $\mathbb{C}$ contains the root of all polynomials, finding explicitly[1] such roots for a given polynomial is a more delicate task. As first noted by Évariste Galois ($\sim$1830), replacing roots by their symmetries –i.e., permutations that respect the original polynomial relations – provides deep insight on the structure of the roots.

Symmetries of roots, a simple example. Look at the polynomial $P = X^4 - 5X^2 + 6$, which factorizes into $P = (X^2 - 3)(X^2 - 2)$, and thus admits exactly the four irrational numbers $\pm\sqrt{2}$ and $\pm\sqrt{3}$ as roots. We shall describe their symmetries as permutation maps[2] on $\{\pm\sqrt{2}, \pm\sqrt{3}\}$. A direct computation shows that there are only four possible such maps as shown in Figure 1.

$$\begin{array}{rll} e: & \sqrt{2} \mapsto \sqrt{2}, & \sqrt{3} \mapsto \sqrt{3}; \\ \varphi_1: & \sqrt{2} \mapsto -\sqrt{2}, & \sqrt{3} \mapsto \sqrt{3}; \\ \varphi_2: & \sqrt{2} \mapsto \sqrt{2}, & \sqrt{3} \mapsto -\sqrt{3}; \\ \varphi_3: & \sqrt{2} \mapsto -\sqrt{2}, & \sqrt{3} \mapsto -\sqrt{3}. \end{array}$$

Fig. 1. Symmetries of the field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$

Let us write $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ for the field generated by $\mathbb{Q}$ and the elements of $\{\pm\sqrt{2}, \pm\sqrt{3}\}$. The four above permutations are called *the symmetries of the roots of P or of the field* $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

For any field K of this type – called *number field*, that is generated by a finite number of irrational numbers – one obtains similarly the so-called *Galois group of symmetries* $\mathrm{Gal}(K/\mathbb{Q})$, that arises as permutations of roots of equations for the associated polynomial. In the example above, the group $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q})$ is $\{e, \varphi_1, \varphi_2, \varphi_3\}$ with a group structure defined by $\varphi_1 \circ \varphi_1 = e$, $\varphi_2 \circ \varphi_1 = \varphi_3$, etc, to form the so called "Klein group", written V_4 – geometrically, this group can also be seen as the group of symmetries of a rectangle.

Symmetries and lattice of subfields. The following result shows that a certain property of the field structure of such a field K can be dealt with via the symmetries of its Galois group.

> **Galois correspondence.** *There is a one-to-one correspondence between the fields that are contained in K, and the subgroups of* $\mathrm{Gal}(K/\mathbb{Q})$.

The example of Figure 2, which is taken from "A Worked out Galois Group for the Classroom"[3], to which we refer for detailed computations and notations, illustrates the potential intricacy of this correspondence in the case where the Galois is the alternate group A_4 – also the group of isometries of a tetrahedron[4].

By recasting an arithmetic question into a group theoretic property, this correspondence allows Galois to show that *not every polynomial equation of degree bigger than* 5 *has*

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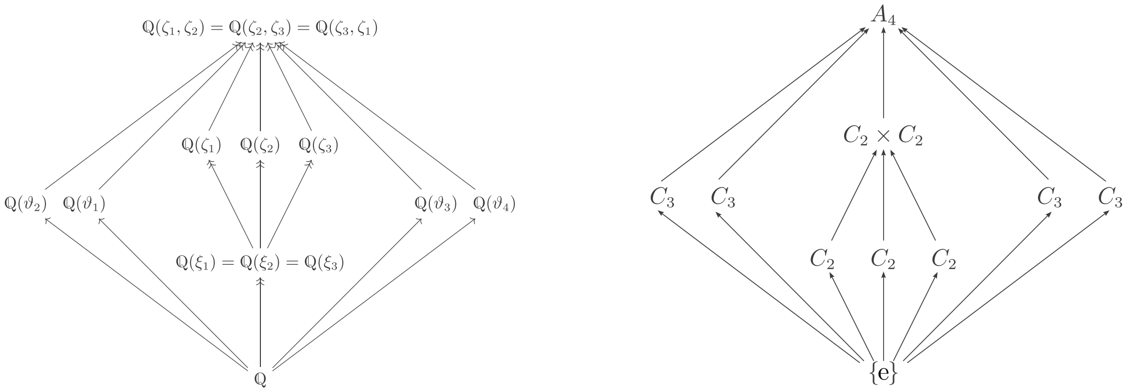
[1]Via an algorithm that uses the four elementary operations and the extraction of roots.

[2]More precisely, such a permutation maps ϕ is defined *on the field* $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ such that (i) for any $a \in \mathbb{Q}$ it holds that $\phi(a) = a$, for any α, $\beta \in K$ it holds (ii) that $\phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta)$ and (ii') that $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$.

[3]By L. Halbeisen and N. Hungerbühler, in The American Mathematical Monthly, 131 (6), p. 501–510. DOI – Use of figures under CC BY 4.0

[4]The reader can draw the Galois correspondence for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, which is much more simple.



(A) Fields Lattice (B) Groups Lattice

FIG. 2. Galois correspondence for $P = X^6 - 3X^2 - 1$

computational solutions. Because the inclusion lattice is preserved, this result is also our first example[5] of field information reconstruction from their symmetries:

> **First reconstruction.** *The symmetries, as encoded by* $\mathrm{Gal}(K/\mathbb{Q})$ *determine the lattice of all the subfields of* K.

See again the example of Figure 2.

A Modern Glimpse. *The reciprocal question, known as the Galois inverse problem which was proposed by Hilbert in* 1892*, to know if* every finite group can be obtained as a Galois group over the rational numbers *is still unresolved and continues, also with its variant the "Noether problem", to stimulate and to shape contemporary research – see Olivier* WITTENBERG*'s [Wit18].*

1.2. **...to reconstructions for numbers and spaces.** The partially successful considerations of *finite* symmetries for numbers and the analogy with *geometric* symmetries motivate to investigate the problem of *the existence of a more canonical and unifying context for reconstruction.*

The field structure of numbers. In the case of numbers, we can define the field $\overline{\mathbb{Q}}$ of all algebraic numbers contained in $\mathbb{C}$, which thus contains all the root of all polynomials. It contains all the towers of all finite extensions $\mathbb{Q} \subset K_1 \subset \cdots \subset K_n \subset \cdots \subset \bar{\mathbb{Q}}$, where for example $K_1 = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ as in the previous section.

One can then form *the absolute Galois group* of rational numbers $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which is the seed of number theory, and whose structure[6] is quite rich. While, in terms of elements, much remains unknown about $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, in terms of symmetries, an analogue of the fundamental theorem of Galois theory holds: *any field* K *between* $\mathbb{Q}$ *and* $\overline{\mathbb{Q}}$ *corresponds to one and only one (closed) subgroup of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

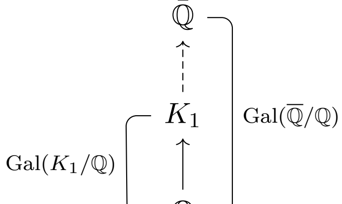


FIG. 3. Tower of fields

[5]The Class Field Theory – developed by Hilbert, Takagi and Artin between 1880 and 1920 – which establishes a correspondence between (the abelianization of) a Galois group and certain data attached to the number field, can be seen as another ancestor to anabelian geometry of Section 2

[6]The absolute Galois group of rational numbers is an infinite topological group.

It is remarkable that, in this context, and as established in successive steps by Jürgen NEUKIRCH, IKEDA Masatoshi, KOMATSU Keiichi, IWASAWA Kenkichi, and UCHIDA Kôji (∼1970), one can go beyond the reconstruction of the field lattice of the previous section:

> **Galois reconstruction of number fields.** *There exists an algorithm which, starting from a group G of the type of an absolute Galois group of a number field, gives the reconstruction of the number field K.*

The Galois symmetries thus encode all the information of K and its field structure. Note that this algorithmic result, that involves only one group (and not the comparison of two), is a contemporary refinement of HOSHI Yuichiro in [Hos22] of the original result.

A geometric recasting. Let us see how the group of Galois symmetries for numbers admits an equivalent for geometric spaces[7], that is the group of loops on the space.

Loops on a topological space X (over $\mathbb{C}$) are just continuously deformable closed curves (with same starting and ending point $*$), which can be composed by concatenation to form a group called the topological fundamental group, written $\pi_1^{top}(X, *)$. If we consider a (finite) covering of X by another manifold X_1, as in Figure 4, we notice that a loop γ on X defines, by sending the starting point x to the end point $\phi(x)$, a transformation ϕ of X_1. The corresponding group of transformations $\mathrm{Aut}(X_1/X)$ is the analog of Galois symmetries $\mathrm{Gal}(K_1/\mathbb{Q})$ for spaces.

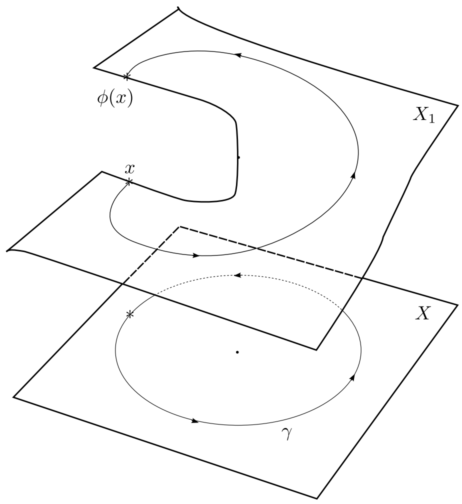


FIG. 4. Symmetries: a loop and a covering.

This thus allows us to view the fundamental group as a representation of symmetries, which raises the question to know *if geometric spaces can also be determined by their symmetries.* In the case of manifold that are locally saddle-like, or *hyperbolic*, one has:

> **Mostow's rigidity theorem (∼1968).** *The topological fundamental group of a hyperbolic manifold of dimension greater than* 2 *completely determines the manifold.*

We thus obtain two separated reconstructions from symmetries: *in arithmetic* with the Neukirch-Ikeda-Iwasawa-Uchida+ theorem , and *in geometry* with Mostow's rigidity theorem. As we will see in the next section, the unifying context for our reconstruction is provided by Grothendieck's arithmetic geometry.

A Modern Glimpse. *An extension of the original (non-algorithmic) Neukirch-Ikeda-Iwasawa-Uchida result was extended by Florian POP (1994). More recently, it was shown by TAMAGAWA Akio and Mohamed SAÏDI that only a smaller portion of the Galois group is sufficient to reconstruct the field – the "m-step solvable Neukirch-Uchida theorem" – see [ST22] and also [Pop21].*

[7] Or *manifolds*, spaces, in our case over $\mathbb{C}$, which can be of any dimension and are obtained as gluing pieces of (our) usual euclidean spaces, and which can have holes and doughnut-like shapes.

2. The universality of homotopic arithmetic geometry

It follows Alexander Grothendieck's vision[8] that the unifying context of arithmetic and geometry, or "homotopic arithmetic geometry", is provided by algebraic varieties and their étale fundamental groups.

2.1. Étale reconstructions for algebraic varieties. An algebraic variety X defined over a field K is the analog of a manifold obtained by patching the zero loci of polynomial equations with coefficients in K – we refer to Figure 5 for an example of a singular curve[9] over $K = \mathbb{C}$.

Similarly to the previous section, its *étale fundamental group* $\pi_1^{\text{et}}(X, *)$ encodes the transformations – or étale symmetries – of certain types of coverings. The étale fundamental group is a profinite topological group, which, since one has the following identifications

- for a manifold X over $\mathbb{C}$: $\pi_1^{\text{et}}(X, *) = \hat{\pi}_1^{\text{top}}(X, *)$,
- for a number field: $\pi_1^{\text{et}}(\operatorname{Spec} K) = \operatorname{Gal}(\bar{K}/K)$,

generalizes both the absolute Galois group and the topological fundamental group.

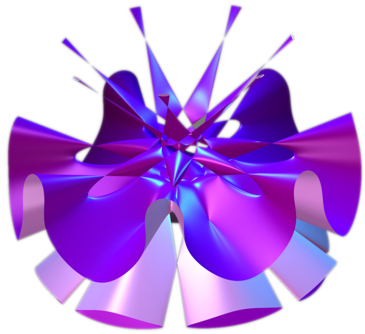


Fig. 5. An algebraic surface of degree 7

When the arithmetic meets the geometry. By standing in the following exact sequence – where X^{an} denotes the complex variety defined by the polynomial equations of X with solutions taken in $\mathbb{C}$ instead of K:

$$\text{(FES)} \qquad 1 \to \underbrace{\hat{\pi}_1^{\text{top}}(X^{\text{an}}, *)}_{\text{Geometry}} \to \pi_1^{\text{et}}(X, *) \to \underbrace{\operatorname{Gal}(\bar{K}/K)}_{\text{Number theory}} \to 1,$$

the étale fundamental group *intermingles number theory and geometry.*

The sequence (FES) above further defines some actions of the absolute Galois group on (a version of) the topological fundamental group: when X is the complex plane with 2 points taken out (whose fundamental group is generated by one loop around each point), the action of $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is computable and is the origin of encoding $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ via the geometric combinatoric of spaces – the so-called Galois-Teichmüller theory[10], see [Oes03] for an introduction.

Anabelian arithmetic geometry, then... Anabelian[11] arithmetic geometry deals with the inverse process of the previous section, that is the question of the reconstruction of spaces from exact sequences of the type of (FES). This kind of geometry emanates from a 1987 conjecture of Grothendieck, which, in its original form, states that *any isomorphism between*

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[8] As exposed in his foundational letter to Gerd Faltings, see [Gro97].

[9] A one-dimensional complex variety, or curve, appears as a two-dimensional real variety, or surface. Picture entitled "Labs Septic" by Oliver Lab, in "IMAGINARY – through the eyes of mathematics", see `https://www.imaginary.org/gallery/oliver-labs` for details and polynomial equation.

[10] This approach is traditionally called "Geometric Galois action"; one uses the terms Galois- or Grothendieck-Teichmüller theory whether one emphasizes the arithmetic or combinatorial nature of the theory. It provides key inputs for the reconstruction of varieties from their étale symmetries.

[11] Contrary to a first idea "Anabelian" does not stand for "ana-belian", from the Greek "ana=anew", but for "an-abelian" with "an=without" and "abelian=commutative" – from the mathematician Abel. Anabelian geometry deals with objects whose fundamental group lacks of commutativity.

the étale fundamental groups of two hyperbolic curves comes from a unique isomorphism between the varieties themselves.

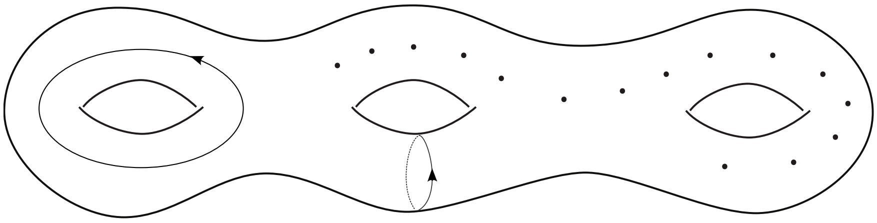

FIG. 6. A hyperbolic curve of genus 3 with r marked points

This conjecture has been resolved by successive progress, which each introduces decisive and lasting arithmetic insights – we refer to the still pertinent and illuminating survey [NTM01] for more details and additional comments:

(1) *Genus* 0 *curves over number fields.* NAKAMURA Hiroaki (1991)
With the use of Deligne weight theory to identify genus and number of marked points of the curves, and the reconstruction of inertia groups (or loops around the marked points) which are generators of the étale fundamental group (for any genus).

(2) *Affine curves, any genus, and over number fields.* TAMAGAWA Akio (1997)
With the use of Kummer theory, which reconstructs multiplicative first (and additive structure then) and will lead to the notion of Kummer-faithfulness for the base field.

(3) *Any curves and genus, and over p-adic fields*[12]*.* MOCHIZUKI Shinichi (1999)
With the use of p-adic Hodge theory, of line bundles instead of coverings, and the principle of container to reconstruct the curve, which will lead to the contemporary mono-anabelian question.

At this stage, anabelian geometry *exploits or reformulates some classical techniques from:*

- *Number theory:* local and global class field theory,
- *Algebraic geometry:* Deligne theory of weights, an anabelian good reduction criterion à la Néron–Ogg–Shafarevich, and the Lefschetz trace formula to isolate rational points in covers.

It also *extends fundamental previous results* such as: the Neukirch-Ikeda-Iwasawa-Uchida theorem (from dimension 0, a field, to dimension 1, a curve) or Faltings' isogeny theorem (from abelian to non-abelian groups) which is a key step in establishing the finitness of rational points on curves (the Mordell conjecture) – see ibid.

2.2. **... and now: contemporary anabelian developments.** Beyond Grothendieck's (relative) initial vision, anabelian geometry now focuses on *establishing and exploiting the universality of arithmetic homotopy*[13], that is, how algebraic spaces can be studied via the

[12] A p-adic field $K/\mathbb{Q}_p$ can be thought of as a neighborhood of a prime p in $\mathbb{Q}$. It must be noted that Grothendieck original conjecture for K a p-adic field does not hold (but a refined version does): there exists non-isomorphic p-adic fields with isomorphic Galois groups.

[13] As opposed, for example, to number theory (that is too rigid and "does not see enough") or complex algebraic geometry (that is too flexible and "sees too much").

canonical group-theoretic properties of their étale fundamental groups. A panorama of recent progress suggests the two following questions:

Universality of anabelian arithmetic geometry.

(1) *Which kind of new insight, in number theory and algebraic geometry, is given by the anabelian homotopic method?*

(2) *Does there exist a group-theoretic algorithm, which from a group that is of étale fundamental group type for a hyperbolic curve, reconstructs the original hyperbolic curve?*

These structuring questions illustrate, respectively, *the ubiquity* and *the canonicity* of the homotopic method in arithmetic geometry. In return, they both lead to beautiful mathematical insights and interactions – we refer to [Boy25] for a living exchange between Hoshi, Mochizuki, Tamagawa, and the first author on this topic.

A question from the attentive reader. *Before these considerations, the attentive reader naturally, and more prosaically, has certainly wondered:* "After the dimension 0 and the dimension 1... is there, similarly to Mostow's theorem, some anabelian reconstructions in higher dimension[14]?" *One difficulty comes here from the so-called "purity property" of the étale fundamental group, which does not see any information in subspaces of small dimension (i.e, of codimension at least* 2*). The technique of polycurves and quasi-tripods has nevertheless allowed Hoshi Yuichiro to establish that* (smooth) algebraic varieties of any dimension possess a fundamental system of anabelian neighborhood*, another conjecture of Grothendieck – see Hoshi's [Hos21], also Schmidt-Stix [SS18].*

Canonicity: algorithm and new structures. The canonicity of anabelian geometry is mostly expressed in the consideration of algorithms that, in the construction steps, eliminate any choice and rely on group-theoretic arguments only – see the sketch[15] given in Algorithm 1, and more generally [Hos25] for a broader and recent survey.

Algorithm 1 Anabelian reconstruction for a hyperbolic curve of strict Belyǐ type

1: **Geometric group:** The maximal finitely generated normal subgroup of $\Pi \rightsquigarrow \Delta$
2: **Genus and points:** Vector space dimension in weighted étale cohomology $\Delta \rightsquigarrow (g, r)$
3: **Inertia groups:** By Belyǐ cuspidalization, the decomposition, then inertia groups $(\Delta, r) \rightsquigarrow D_x$ and I_x
4: **Multiplicative monoid:** Kummer theory $\rightsquigarrow$ multiplicative monoid $(K(X)^*, \boxtimes)$
5: **The (function field of) X:** Uchida and divisors $\rightsquigarrow (K(X), \boxtimes, \boxplus)$
6: **The base field of X:** $K(X) \rightsquigarrow K_X$

Here Π (resp. Δ) denotes a group isomorphic to a certain $\pi_1^{\text{et}}(X, *)$ (resp. $\hat{\pi}_1^{\text{top}}(X^{\text{an}}, *)$).

This approach in particular reveals the importance of **two types of new structures**: **(a)** a distinction between "étale-like" and "Frobenius-like" objects, where the former is used to transport the rigid information of the latter – see the mono-anabelian transport of Hoshi in [Hos22]; and **(b)** the seminal role of multiplicative monoids, a structure that is *weaker than the ring one, on which is initially based both Galois theory and Grothendieck algebraic geometry.*

[14]Two other directions are given by the investigation of arithmetic phenomenons (a) in higher homotopy and (b) in higher categories – we refer respectively to Schmidt and Stix' [SS18] and to the first author's [Col21].

[15]We must insist that this algorithm applies to very specific types of curves only, and that the algorithm, not the resulting isomorphism, is important.

In another direction, the **combinatorial anabelian geometry** of Hoshi and Mochizuki, can be seen as a functorial version of the Galois-Teichmüller theory for braids groups of Section 2. One obtains, as a result, the anabelianity of Galois-Teichmüller theory and a promising combinatorial model of the algebraic closure $\bar{\mathbb{Q}}$ of the rational numbers – see Tsujimura Shota et al. in [Tsu23], and [Phi24] for a structured introduction with references.

The **nearly-abelian or minimalistic program** – also the m-step solvable Grothendieck conjecture – that investigates how "anabelianly big" the étale fundamental group must be to reconstruct the spaces, can be seen as a variant of the canonicity property – see Tamagawa-Saïdi for Galois groups, the geometric version of the third author [Yam24], the Pop's Neukirch-Uchida [Pop21], and the Galois-Teichmüller approach of Adam Topaz in [Top21].

Universality: ramifications and explicit methods. The first example of the anabelian universality, can be seen in the result that a certain Grothendieck-Teichmüller group[16] stemming from the Galois-Teichmüller theory of Section 2.1, is indeed anabelian – in 2017 by Hoshi, Minamide and Mochizuki, see [Tsu23]. In another direction, **Berkovich geometry** has played a key role in connecting Tamagawa's resolution of non-singularities to establish, in its abolute version, the Grothendieck conjecture *over p-adic fields* – see [Lep23]. The anabelian method applied to **the reconstruction of theta functions of elliptic curves** with application to establishing Diophantine height inequality open new unsuspected perspectives – see [Moc23] and [Col24] for a first contact and references. More recently, one observes new connections (e.g., in relation to the anabelian "Kummer faithfulness") with **classical Galois theory *à la* Field Arithmetic** – see the second author's [Mur23], Sawada et al. in [MST24], and Asayama-Taguchi in [AT24].

Explicit methods mostly follow the "Galois-Teichmüller thread" (now known to be anabelian, see above). Among many, we can refer only briefly to recent developments[17] of **Ihara's program in number theory** – see "When does Yama meets Ten?"[18] pursued by Rasmussen-Tamagawa 2017-... in [MR24], or interaction with Greenberg's program by Pries; or in relation with **motivic theory** – via Deligne-Ihara Lie algebra, see Ishii; or in relation with **low-dimensional topology** – such as Oda's prediction (see Philip) and Morita obstruction (Nakamura et al., see list of references ibid.) – all references taken from the report of Footnote 17.

Acknowledgments. The last two authors would like to thank the organizers of the RIMS-Oberwolfach workshop "Arithmetic Homotopy and Galois Theory"[19], for the opportunity of writing this Snapshot, as well as all the participants for the enriching and stimulating mathematical atmosphere.

[16]This GT group is related to diverse mathematical fields , such as, motivic theory, braid and Teichmüller mapping class groups, operads, and quasi-triangular quasi-Hopf algebra.

[17] See the corresponding reports in the already cited proceedings "MFO–RIMS Tandem Workshop: Arithmetic Homotopy and Galois Theory (B. Collas, P. Dèbes, Y. Hoshi, and A. Mézard, eds.), vol. 20, Oberwolfach Rep., EMS Press, 2023".

[18]"When does the mountain meet the heaven?"

[19]Jointly held at the Research Institute for Mathematical Sciences, Kyoto University and the Mathematisches Forschungsinstitut Oberwolfach from 24-29 September 2023, see web page at https://ahgt.math.cnrs.fr/activities/workshops/MFO-RIMS23/.

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