Lecture RIMS - Kyoto University

HOMOTOPICAL ARITHMETIC GEOMETRY OF MODULI STACKS OF CURVES

2018, Oct.1 - 15

Abstract: Moduli spaces of curves are ideal spaces for studying fundamental abstract theories of arithmetic geometry: they give geometric Galois representations that can be explicitly computed, furnish examples of anabelian spaces, and in genus zero generate the category of mixed Tate motives. They also possess a dual nature, being either considered as schemes or algebraic stacks.
The goal of this series of talks is to provide a basic introduction to these aspects by covering various fundamental geometric and arithmetic properties. It is intended for graduate students in algebraic geometry and non-specialists researchers. Elementary notions will be either recalled or illustrated with pictures or examples.

ALGEBRAIC & DELIGNE-MUMFORD STACKS — LECTURES 1 AND 2

Taking the functor of points for schemes as initial motivation, we introduce the notion of stacks as lax functors in groupoids with descent conditions and show how to recover Laumon-Moret-Bailly's original definition. We present how the Artin and Deligne-Mumford algebraic versions – that admit topological coverings by schemes – allow to "push" algebraic geometry properties in this context.

Keywords: diagrams of groupoids, Grothendieck topologies, examples of global quotient and inertia stacks.

REFERENCES

[Ber13] J. Bertin. "Algebraic stacks with a view toward moduli stacks of covers". In: *Arithmetic and geometry around Galois theory*. Vol. 304. Progr. Math. Birkhäuser/Springer, Basel, 2013, pp. 1–148.

[Ols16] M. Olsson. *Algebraic spaces and stacks*. Vol. 62. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2016, pp. xi+298.

[Stacks] T. Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`. 2018.

MODULI PROBLEMS & MODULI SPACES OF CURVES — LECTURE 3

We present how the scheme-stack structures and the geometry of curves lead to two solutions for building classifying spaces. Having introduced the notion of functor of moduli, we present Gieseker and Deligne-Mumford constructions of the moduli space of stable curves: the former follows Mumford GIT-theory and gives a projective scheme, the latter produces a smooth algebraic Deligne-Mumford global stack.

Keywords: Hilbert scheme, explicit examples in low genus, stable compactification, formal neighbourhood.

REFERENCES

[DM69] P. Deligne and D. Mumford. "The irreducibility of the space of curves of given genus". In: *Publications Mathématiques de l'IHES* 36.1 (1969), pp. 75–109.

[Gie82] D. Gieseker. *Lectures on moduli of curves*. Vol. 69. Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Springer-Verlag, Berlin-New York, 1982, pp. iii+99.

[Knu83] F. F. Knudsen. "The projectivity of the moduli space of stable curves. II. The stacks $\mathrm{M}_{g,n}$". In: *Math. Scand.* 52.2 (1983), pp. 161–199.

[GIT] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*. Third. Vol. 34. Ergebnisse der Mathematik und ihrer Grenzgebiete (2). Springer-Verlag, Berlin, 1994, pp. xiv+292.

Fundamental Group & Arithmetic — Lecture 4

We follow Grothendieck construction of the étale fundamental group that leads to Geometric Galois actions of the absolute Galois group of rational on the geometric fundamental group of moduli stack of curves. We adapt this approach in the case of Deligne-Mumford stacks and show how it leads to a divisorial and a stack arithmetic of the spaces. Following the seminal work of Ihara, Matsumoto and Nakamura, we present explicit results and properties of the former, then recent similar results in the case of cyclic inertia for the latter.

Keywords: étale fundamental group for stacks, explicit computations in low dimensions, tangential Galois representations.

References

[CM14] B. Collas and S. Maugeais. *On Galois action on stack inertia of moduli spaces of curves.* 2014. arXiv: `1412.4644 [math.AG]`.

[SGA1] A. Grothendieck. *Revêtements étales et groupe fondamental.* Documents mathématiques. Société Mathématique de France, 2003.

[IN97] Y. Ihara and H. Nakamura. "On deformation of maximally degenerate stable marked curves and Oda's problem". In: *J. Reine Angew. Math.* 487 (1997), pp. 125–151.

[Nak97] H. Nakamura. "Galois representations in the profinite Teichmüller modular groups". In: *Geometric Galois actions, 1.* Vol. 242. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1997, pp. 159–173.

[NS00] H. Nakamura and L. Schneps. "On a subgroup of the Grothendieck-Teichmüller group acting on the tower of profinite Teichmüller modular groups". In: *Inventiones mathematica* 141.141 (2000), pp. 503–560.

[Noo04] B. Noohi. "Fundamental groups of algebraic stacks". In: *Journal of the Institute of Mathematics of Jussieu* 3.01 (2004), pp. 69–103.

Motivic Theory for Moduli Stack of Curves — Lecture 5

We present recent progress on an ongoing project on the construction of a category of motives for the moduli stacks of curves, whose main property is to reflect the arithmetic properties of the cyclic stack inertia. Having recalled briefly some already available categories of Chow, Grothendieck and Voevodsky (derived) motives, we first present Morel-Voevodsky stable/unstable motivic homotopy categories. We then show how the homotopical-simplicial approach is compatible with Friedlander étale topological type and is well adapted to our goal.

The various model categories hidden in the previous lectures are revealed.

Keywords: Quillen model category, Artin-Mazur étale topological type, Mixed Tate motives and loop space.

References

[Col18] B. Collas. "Homotopy and Arithmetic Geometry of Moduli Stack of Curves". In: *Mini-Workshop: Arithmetic Geometry and Symmetries around Galois and Fundamental Groups.* OWR Report. 2018.

[Cox79] D. A. Cox. "Homotopy theory of simplicial schemes". In: *Compositio Math.* 39.3 (1979), pp. 263–296.

[Dun+07] B. I. Dundas et al. *Motivic homotopy theory.* Universitext. Lectures from the Summer School held in Nordfjordeid, August 2002. Springer-Verlag, Berlin, 2007, pp. x+221.

[Lev13] M. Levine. "Six lectures on motives [Lectures 3 and 5]". In: *Autour des motifs—École d'été Franco-Asiatique de Géométrie Algébrique et de Théorie des Nombres/Asian Vol. II.* Vol. 41. Panor. Synthèses. Soc. Math. France, Paris, 2013, pp. 1–141.

[Oda97] T. Oda. "Etale homotopy type of the moduli spaces of algebraic curves". In: *Geometric Galois actions, 1.* Vol. 242. London Math. Soc. Lecture Note Ser. Cambridge: Cambridge Univ. Press, 1997, pp. 85–95.

[Voe98] V. Voevodsky. "$\mathbf{A}^1$-homotopy theory". In: *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998).* Extra Vol. I. 1998, pp. 579–604.

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