Notes on
Galois group and Grothendieck-Teichmüller theory

Action on torsion-elements of $\pi_1^{geom}(\mathcal{M}_{0,[n]})$ by cohomological methods

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Abstract
We follow Grothendieck's idea to study $G_{\mathbb{Q}}$ using its geometric action, and we obtain the expression of this action on torsion elements of the geometric fundamental group $\pi_1^{geom}(\mathcal{M}_{0,[n]})$.

$$\begin{array}{ccc} G_{\mathbb{Q}} & \hookrightarrow & \mathrm{Aut}(\pi_1^{geom}(X)) \\ & \searrow & \nearrow \\ & \widehat{GT} & \end{array}$$

Questions

- which X to choose ?
- how $\widehat{GT}$ can help ?

1 Galois and Grothendieck-Teichmüller groups

1.1 Galois geometric representation

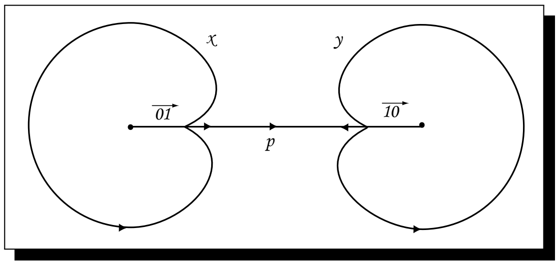
The Fundamental Exact Sequence
Let X be a smooth absolutely irreducible variety over $\mathbb{Q}$.

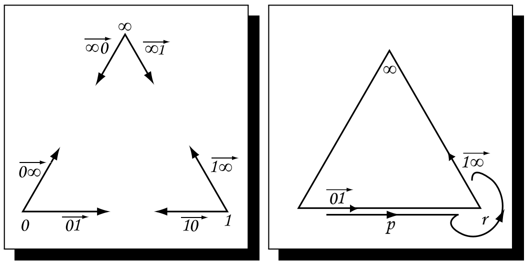
$$1 \longrightarrow \pi_1^{alg}(X \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}) \longrightarrow \pi_1^{alg}(X) \underset{\overset{s}{\longleftarrow}}{\longrightarrow} G_{\mathbb{Q}} \longrightarrow 1 \qquad \text{(FES)}$$

where $\pi_1^{alg}(X \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}) \simeq \widehat{\pi}_1^{top}(X(\mathbb{C}))$.

This leads to a geometric representation. And once a section s, ie a $\mathbb{Q}$-rational point, is chosen,

$$G_{\mathbb{Q}} \to \mathrm{Aut}(\widehat{\pi}_1^{top}(X(\mathbb{C}))$$
$$\sigma \mapsto (\gamma \mapsto \sigma\gamma\sigma^{-1})$$





Key example: $X_4 = \mathbb{P}^1\overline{\mathbb{Q}} - \{0, 1, \infty\}$
Since $\pi_1^{top}(X_4) = \mathbb{F}_2$, once a rational base point chosen, the (FES) gives, by a theorem of *Belyi*:

$$G_{\mathbb{Q}} \hookrightarrow \mathrm{Aut}(\widehat{\mathbb{F}}_2)$$

1.2 Projective line minus 3 points

Theorem A
There exists a faithful action

$$\begin{array}{ccc} G_{\mathbb{Q}} & \hookrightarrow & \mathrm{Aut}(\widehat{\mathbb{F}}_2) \\ \sigma & \mapsto & \phi_\sigma \end{array}$$

defined, where $\chi(\sigma)$ is the usual cyclotomic character and $f_\sigma \in \widehat{\mathbb{F}}_2'$, by

$$\phi_\sigma(x) = x^{\chi(\sigma)}, \quad \phi_\sigma(y) = f_\sigma^{-1}(x, y) y^{\chi(\sigma)} f_\sigma(x, y).$$

To avoid breaking the symmetries, consider:

- $Aut(X_4) = S_3 =< \omega, \theta >$ where

$$\theta : t \mapsto 1 - t \qquad \omega : t \mapsto (1 - t)^{-1}$$

- The 6 tangential base points of X_4

$$\mathcal{B} = \{\overrightarrow{01}, \overrightarrow{10}, \overrightarrow{1\infty}, \overrightarrow{\infty 1}, \overrightarrow{\infty 0}, \overrightarrow{0\infty}\}$$

- The fundamental topological groupoïd $\pi_1^{top}(X_4, \mathcal{B})$ compound with paths from a tangential base point to another, modulo homothopy.

Explicit action on x and y

- Action on x is defined by monodromic action on Puiseux series.

$$x.(\sum_n a_n t^{\frac{n}{k}}) \to \sum_n a_n \zeta^n t^{\frac{n}{k}} \text{ where } \zeta = exp(2i\pi/k)$$
$$\sigma.x.(\sum_n a_n t^{\frac{n}{k}}) \xrightarrow{\sigma^{-1}} \sum_n \sigma^{-1}(a_n) t^{\frac{n}{k}} \xrightarrow{x} \sum_n \sigma^{-1}(a_n)\zeta^n t^{\frac{n}{k}} \xrightarrow{\sigma}$$
$$\xrightarrow{\sigma} \sum_n \sigma(\sigma^{-1}(a_n)\zeta^n) t^{\frac{n}{k}} = \sum_n a_n \zeta^{n\chi(\sigma)} t^{\frac{n}{k}} = x^{\chi(\sigma)}.(\sum_n a_n t^{\frac{n}{k}}).$$

- Action on y comes from symmetries of X_4. Let $p \in \pi_1(X_4; \overrightarrow{01}, \overrightarrow{10})$ and $\theta : t \mapsto 1-t \in Aut(X_4)$. Then
$$y = p^{-1}\theta(x)p.$$
A similar computation on y makes appear the following element
$$f_\sigma(x,y) := p^{-1}\sigma p \in \widehat{\mathbb{F}}'_2.$$

Theorem B
We have a parametrization:
$$\begin{array}{ccc} G_{\mathbb{Q}} & \to & \widehat{\mathbb{Z}}^* \times \widehat{F}'_2 \\ \sigma & \mapsto & (\chi_\sigma, f_\sigma) \end{array}$$
Where, f_σ satisfies the following two equations:

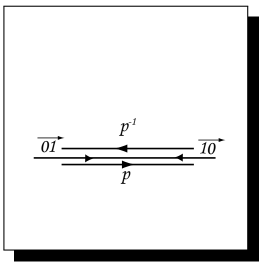
(I) $\quad f_\sigma(x,y)f_\sigma(y,x) = 1$

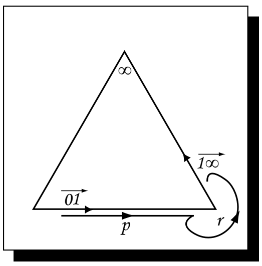
(II) $\quad f_\sigma(z,x)z^{m_\sigma}f_\sigma(y,z)y^{m_\sigma}f_\sigma(x,y)x^{m_\sigma} = 1 \quad$ where $z = (xy)^{-1}$and $m_\sigma = \frac{1}{2}(\chi_\sigma - 1)$.

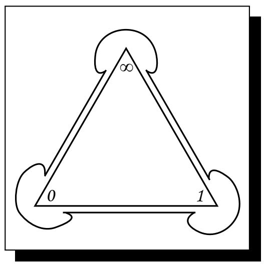
These 2 relations come from the essential symmetries of X_4:

(I) is obtained by applying σ on the geometric relation, where $p \in \pi_1(X_4; \overrightarrow{01}, \overrightarrow{10})$
$$\theta(p)p = 1 \qquad f_\sigma(x,y)f_\sigma(y,x) = 1$$

(II) Let $q = r \circ p \in \pi_1(X_4; \overrightarrow{01}, \overrightarrow{1\infty})$ where $r \in \pi_1(X_4, \pi_1(X_4; \overrightarrow{10}, \overrightarrow{1\infty})$ and $\omega \in Aut(X_4)$. Then
$$\omega^2(q)\omega(q)q = 1 \qquad \text{implies } (II)$$







1.3 Grothendieck-Teichmüller group

Definition of $\widehat{GT}$

Starting from the previous parametrization:

$$\sigma \in G_{\mathbb{Q}} \hookrightarrow (\chi_\sigma, f_\sigma) \in \hat{\mathbb{Z}}^\times \times \mathbb{F}_2'.$$

One can try to define an internal composition law. Lets $\sigma, \tau \in G_{\mathbb{Q}}$, then

$$\begin{aligned} x &\xrightarrow{\tau} x^{\chi_\tau} \xrightarrow{\sigma} \qquad x^{\chi_\tau \chi_\sigma} \\ y &\xrightarrow{\tau} f_\tau^{-1} y^{\chi_\tau} f_\tau \xrightarrow{\sigma} F_\sigma^{-1}(f_\tau) f_\sigma^{-1} y^{\chi_\tau \chi_\sigma} f_\sigma F_\sigma(f_\tau) \end{aligned}$$

And eventually

$$(\chi_\sigma, f_\sigma).(\chi_\tau, F_\tau) = (\sigma\tau, f_\sigma F_\sigma(f_\tau))$$

Using the previous symmetries in X_4 (and a bit more) yields to (originally defined by Drinfel'd)

Definition

Lets define $\widehat{GT}$ as the group of elements $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{\mathbb{F}}_2'$ such that $(x, y) \mapsto (x^\lambda, fyf^{-1})$ induces an automorphism of $\widehat{\mathbb{F}}_2$ and that satisfy

(I) $f(x, y)f(y, x) = 1$

(II) $f(z, x)z^m f(y, z)y^m f(x, y)x^m = 1$ where $m = (\lambda - 1)/2$

(III) $\tilde{f}(x_{34}, x_{45})\tilde{f}(x_{51}, x_{12})\tilde{f}(x_{23}, x_{34})\tilde{f}(x_{45}, x_{51})\tilde{f}(x_{12}, x_{23}) = 1$ (where $\tilde{f}$ is the image of f in $\Gamma_{0,5}$).

2 Genus 0 Moduli spaces of curves

Strategy

Which "good" geometric space (or category of spaces) for X can we choose to generalize the symmetries found in X_4 in order to:

- create new group and capture some fundamentals properties of $G_{\mathbb{Q}}$,
- obtain a less theoritical and more computable representation.

2.1 Moduli spaces and mapping class group

Let $S_{g,n}$ be a topological space of genus g with n marked points $(x_1, \cdots, x_n)$.

Definition

Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces of genus g with n marked points. Or equivalently, the space of analytic structures on $S_{g,n}$ modulo isomorphism.

- Example: $\mathcal{M}_{0,n} = (\mathbb{P}^1 - \{0, 1, \infty\})^{n-3} - \Delta$ is an algebraic variety. Note that the previous $\mathcal{M}_{0,4} = X_4$.

- As $\mathrm{Aut}(\mathcal{M}_{0,n}) = S_n$, following the previous study of X_4, one considers the unordered $\mathcal{M}_{0,[n]} = \mathcal{M}_{0,n}/S_n$ which is a topological orbifold or a $\mathbb{Q}$-stack.

  Then

  $$\Gamma_{0,n} := \pi_1^{top}(\mathcal{M}_{0,n}) \qquad \Gamma_{0,[n]} := \pi_1^{orb}(\mathcal{M}_{0,[n]}) \text{ as orbifold}$$
  $$\pi_1^{geom}(\mathcal{M}_{0,n}) = \widehat{\Gamma}_{0,n} \text{ as algebraic variety} \qquad \pi_1^{geom}(\mathcal{M}_{0,[n]}) = \widehat{\Gamma}_{0,[n]} \text{ as stack.}$$

From the Fundamental Exact Sequence applied to the fundamental groupoïd $\pi_1(\mathcal{M}_{0,[n]}, \mathcal{B})$ where $\mathcal{B}$ is the set of tangential base points in $\mathcal{M}_{0,[n]}$, it follows

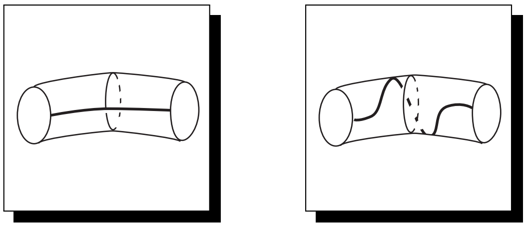
Theorem
For $n \geqslant 4$, there exists an embedding

$$G_{\mathbb{Q}} \hookrightarrow \mathrm{Aut}(\widehat{\Gamma}_{0,[n]}).$$

Mapping class groups as diffeomorphism of surfaces
Paths in $\mathcal{M}_{0,n}$ between two points are continuous deformation of the analytic structures, ie

$$\Gamma_{0,n} = Diff^+(S_{0,n})/Diff^0(S_{0,n}) \qquad \Gamma_{0,[n]} = Diff^+(S_{0,[n]})/Diff^0(S_{0,[n]})$$

Theses groups are generated by Dehn twists along curves:



Theorem [Dehn]

- The pure mapping class group $\Gamma_{0,n}$ is generated by Dehn twists along loops avoiding marked points.
- The full mapping class group $\Gamma_{0,[n]}$ is generated by Dehn twists along loops avoiding marked points or going through 2 marked points (also called half-twist).

2.2 Mapping class group and Artin braids

Definition
The Artin braid groups B_n is generated by the $n-1$ generators $\sigma_1, \cdots, \sigma_{n-1}$ with relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1 \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

The pure Artin braid group $K_n \subset B_n$ is generated by $x_{ij} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1}$

Proposition
Let γ_i a simple loop going through the marked points x_i and x_{i+1}, and τ_i the half-twist it defines. There is a morphism $\tau_i \in \Gamma_{0,[n]} \to \sigma_i \in B_n$ which gives isomorphisms

$$\Gamma_{0,[n]} \simeq B_n/ < y_n, \omega_n > \qquad \Gamma_{0,n} \simeq K_n/Z$$

where $\omega_n = (\sigma_1 \cdots \sigma_{n-1})^n$, $y_n = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1^2\cdots\sigma_{n-2}\sigma_{n-1}$ and $Z =< \{x_{1i}x_{2i}\cdots x_{ni}\}_{2\leqslant i\leqslant n} >$.

2.3 $G_{\mathbb{Q}}$, $\widehat{GT}$ and Braids

This approch by Artin braid groups yields to another proof of the previous theoretic result (first etablished by Drinfel'd)

Theorem (Drinfel'd, Ihara and Matsumoto)
There exists a parametrization $G_{\mathbb{Q}} \hookrightarrow Aut(\widehat{B}_n)$ which extends to $\widehat{\Gamma}_{0,[n]} = \widehat{B}_n / < y_n, \omega_n >$ given by

$$\begin{array}{ccl} G_{\mathbb{Q}} & \hookrightarrow & Aut(\widehat{B}_n) \\ \sigma & \mapsto & \sigma_1 \mapsto \sigma_1^{\chi_\sigma} \\ & & \sigma_i \mapsto f(y_i, \sigma_i^2)^{-1} \sigma_i^{\chi_\sigma} f(y_i, \sigma_i^2) \text{ for } 2 \leq i \leq n-1 \end{array}$$

where $y_i = \sigma_{i-1}\sigma_{i-2} \cdots \sigma_1^2 \cdots \sigma_{i-2}\sigma_{i-1}$.

- Proof: by algebraic geometry on configuration spaces, or action on X_4 and group theoretic arguments in $\widehat{B}_n$.

Moreover, the relation (III) defining $\widehat{GT}$

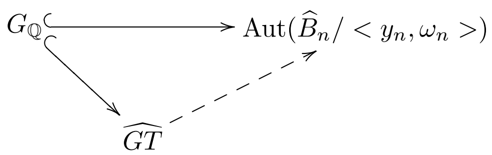
$$(III) \qquad \tilde{f}(x_{34}, x_{45})\tilde{f}(x_{51}, x_{12})\tilde{f}(x_{23}, x_{34})\tilde{f}(x_{45}, x_{51})\tilde{f}(x_{12}, x_{23}) = 1$$

lives in $\widehat{\Gamma}_{0,5}$ ($\widehat{\mathbb{F}}_2 \hookrightarrow \widehat{\Gamma}_{0,5}$) and comes from $G_{\mathbb{Q}}$-action on symmetries of $\mathcal{M}_{0,5}$, such that

Theorem (Ihara-Drinfel'd)
There exists a morphism

$$G_{\mathbb{Q}} \hookrightarrow \widehat{GT}.$$

Following *Lochak and Schneps*, one recovers by braid computations (originally by Drinfel'd)

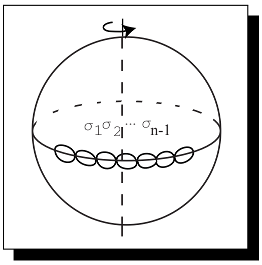


A first conclusion
The category of moduli spaces of curves $\mathcal{M}_{0,[n]}$ is helpfull for

- defining the new group $\widehat{GT}$ which approximates $G_{\mathbb{Q}}$,
- obtaining explicit computations in the braids groups $\widehat{B}_n$.

3 $\widehat{GT}$-$G_{\mathbb{Q}}$ action on torsion

Application
The group $\widehat{GT}$ gives the action of $G_{\mathbb{Q}}$ on torsion elements of $\pi_1^{geom}(\mathcal{M}_{0,[n]})$.



3.1 Torsion elements in $\Gamma_{0,[n]}$

What are the torsion elements in $\Gamma_{0,[n]}$?

Using explicit computations in the Braid group, one can check the following elements have finite order in $\Gamma_{0,[n]}$.

- Order n $\gamma_n = \sigma_1\sigma_2\sigma_3\cdots\sigma_{n-1}$
- Order $n-1$ $\gamma_{n-1} = \sigma_1\sigma_2\sigma_3\cdots\sigma_{n-2}$
- Order $n-2$ $\gamma_{n-2} = \sigma_1^2\sigma_2\sigma_3\cdots\sigma_{n-2}$

Theorem
Let $\gamma \in \Gamma_{0,[n]}$ be a torsion element. Then γ is conjugate to a power of γ_n, γ_{n-1} or γ_{n-2}.

We will now determine the conjugacy classes of the torsion elements in the profinite mapping class group $\widehat{\Gamma}_{0,[n]}$.

3.2 Torsion element in $\widehat{\Gamma}_{0,[n]}$

Some cohomological properties

Proposition
Let $\tau \in \{\gamma_{n-2}, \gamma_{n-1}, \gamma_n\}$ and $\rho_\tau : x \in \widehat{K}_n \mapsto \tau x \tau^{-1}$. The elements of the non-abelian cohomology set $H^1(<\rho_\tau>, \widehat{K}_n/Z)$ correspond to splittings of

$$1 \to K_n/Z \to (K_n/Z) \rtimes <\rho_\tau> \to <\rho_\tau> \to 1$$

up to K_n/Z-conjugacy.

Following Serre.

Definition [Property $(*)$]
Let G be a group and $\{G_i\}_I$ some finite subgroups of G such that:

1. Every finite subgroup of G is conjugate to a finite subgroup of one of $G_1, \cdots, G_r$
2. For $i \neq j$ and $g \notin G_i$, $G_j \cap gG_ig^{-1} = \{1\}$.

Then one says G satisfies the property $(*)$ for the $G_1, \cdots, G_r$.

If $\widehat{\Gamma}_{0,[n]}$ satisfies the property $(*)$ for $G_1 = <\gamma_n>$, $G_2 = <\gamma_{n-1}>$, $G_3 = <\gamma_{n-1}>$, then it is sufficient to determine the set $H^1(<\rho_\tau>, \widehat{\Gamma}_{0,n})$ in order to determine the conjugacy class of the torsion-elements of $\widehat{\Gamma}_{0,[n]}$.

$$\begin{array}{ccc} H^n(G,M) \simeq \prod_1^r H^n(G_i,M) & \xrightarrow{Goodness} & H^n(\widehat{G},M) \simeq \prod_1^r H^n(\widehat{G}_i,M) \\ (*) \text{ implies } (H) \Big\uparrow & & \Big\downarrow (H) \text{ implies } (*) \\ \text{Torsion } \Gamma_{0,[n]} \text{ - } H^1(<\rho>,\Gamma_{0,n}) & & \text{Torsion } \widehat{\Gamma}_{0,[n]} \text{ - } H^1(<\rho>,\widehat{\Gamma}_{0,n}) \end{array}$$

Goodness

We will now reduce the problem to the discrete case using the two following results.

Proposition
Let G be a discrete or profinite group and $G_1, \cdots, G_r$ some finite subgroups of G such that $vcd_p(G) < \infty$ for every prime p. Then for every p-primary G-module M

$$H^n(G,M) \to \prod_1^r H^n(G_i,M) \tag{H}$$

is an isomorphism for $n >> 0$ if and only if G satisfies the property $(*)$ for the G_i.

Definition
A discrete group G is called *good* if it satisfies $H^n(G,M) \simeq H^n(\widehat{G},M)$ for every finite G-module M and every $n \geqslant 0$.

Proposition
The mapping class groups $\Gamma_{0,n}$ and $\Gamma_{0,[n]}$ are *good*.

Torsion elements in $\widehat{\Gamma}_{0,[n]}$

- $\Gamma_{0,[n]}$ satisfies the property $(*)$ for the $G_1 = <\gamma_n>$, $G_2 = <\gamma_{n-1}>$, $G_3 = <\gamma_{n-1}>$.

Proposition
Let ρ_τ as above. Then the two non-commutative sets are equal

$$H^1(<\rho_\tau>,\Gamma_{0,n}) = H^1(<\rho_\tau>,\widehat{\Gamma}_{0,n})$$

As a consequence of the conjugacy classes of the torsion elements in $\Gamma_{0,n}$,

Theorem
Let $\gamma \in \pi_1^{geom}(\mathcal{M}_{0,[n]})$ a torsion element. Then γ is conjugate to a power of γ_n, γ_{n-1} or γ_{n-2}.

3.3 Action of $\widehat{GT}$

Using these results, we can work in the discrete braid groups $\Gamma_{0,[n]}$. By direct computations in B_n we describe explicitely $H^1(*,\Gamma_{0,[n]})$. The $\widehat{GT}$ action on B_n appears then as a ρ_τ-cocycle and one obtains.

Proposition
Let $n \geqslant 1$. For $1 \leqslant i \leqslant n$, the automorphism $F = (\lambda, f) \in \widehat{GT}$ sends the following elements

$$\alpha_i = \sigma_1 \cdots \sigma_i \qquad \beta_i = \sigma_1^2 \cdots \sigma_i \in \widehat{B}_n$$

to $y\alpha^{\lambda}y^{-1}$ and $y\beta^{\lambda}y^{-1}$ for $y \in \widehat{B}_n$ (ie λ-conjugates).

As a consequence,

Theorem
The automorphism $F = (\lambda, f) \in \widehat{GT}$ λ-conjugates all finite-order elements in $\pi_1^{geom}(\mathcal{M}_{0,[n]})$ for $n \geqslant 4$.

And eventually,

Corollary
The galois goup $G_{\mathbb{Q}}$ $\chi(\sigma)$-conjugates the finite-order elements in $\pi_1^{geom}(\mathcal{M}_{0,[n]})$ for $n \geqslant 4$.

Other application of cohomological methods in $\widehat{GT}$

Remark
Using theses cohomological technics with others $\{G_i\}$, *Lochak and Schneps* obtained

- a new cohomological definition of $\widehat{GT}$,
- a new proof of the inclusion $G_{\mathbb{Q}} \to \widehat{GT}$.

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