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Homotopic and Geometric Galois Theory (online meeting)

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Abstract. In his "Letter to Faltings", Grothendieck lays the foundation of what will become part of his multi-faceted legacy to arithmetic geometry. This includes the following three branches discussed in the workshop: the arithmetic of Galois covers, the theory of motives and the theory of anabelian Galois representations. Their geometrical paradigms endow similar but complementary arithmetic insights for the study of the absolute Galois group $G_{\mathbb{Q}}$ of the field of rational numbers that initially crystallized into a functorially group-theoretic unifying approach. Recent years have seen some new enrichments based on modern geometrical constructions – e.g. simplicial homotopy, Tannaka perversity, automorphic forms – that endow some higher considerations and outline new geometric principles. This workshop brought together an international panel of young and senior experts of arithmetic geometry who sketched the future desire paths of homotopic and geometric Galois theory.

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Introduction by the Organizers

Understanding the absolute Galois group $G_{\mathbb{Q}}$ by geometric means is a broad program, which, since Grothendieck, has been developed along three parallel branches: *(a) the Arithmetic of Galois Covers*, *(b) the Motivic Galois Representations*, and *(c) Anabelian Geometry*. The workshop *Homotopic and Geometric Galois Theory* dealt with recent progress around these branches, which in fact showed reuniting

trends *towards a new geometry of Galois symmetries of spaces.* Essential concepts crystallize the existence of cross-bridging principles, which are pushing the seminal programs of the three branches beyond their original frontiers. The specifics of higher dimensions and stacks symmetries act as one joint guiding beacon, the search for unified minimal and functorial anabelian constructions as a second one, confrontation with key seminal objects and results as a third one.

Overview. A previous 2018 Oberwolfach mini-workshop – *Arithmetic Geometry and Symmetries around Galois and Fundamental Groups* – had reported on the most striking developments and connections of the three branches:

(a) the success for supersolvable groups of the Colliot-Thélène approach to the Noether program (Harpaz-Wittenberg), a realizing-lifting-parametrizing program pushing further the Hilbert specialization method (Dèbes, Fried, et al.), Tannakian considerations within $\mathbb{Q}_\ell$-perverse sheaves (Dettweiler et al.), the use of patching methods for local-global issues (Harbater et al.);
(b) the motivic constructions of given Tannaka groups using automorphic and perverse techniques (Yun, Patrikis, Katz), the use of p-adic or ultraproduct techniques (Cadoret, Ambrosi) to study ℓ-adic motives, the characterization of geometric representations in deformation rings (Litt), the proof of a Deligne conjectured fixed-point counting formula (Yu);
(c) the successful introduction of methods from étale homotopy theory (Schmidt-Stix) and from motivic homotopy theory for moduli stacks of curves (Collas), the import of operads (Fresse-Horel), a computable echo of the Galois techniques in the Ihara/Oda-Matsumoto problem (Pop) and of anabelian group-theoretic reconstructions (Hoshi, Mochizuki).

Building on this first round of investigations, the 2021 workshop provided more mature encounters that resulted *in a unifying view of these topics.* With the goal of exploiting new arithmetic-geometric symmetries – in *higher dimension*, in *higher categories* – as a common guide, the set of contributions revealed the hidden desire paths of homotopic and geometric Galois theory: there is *a return from Grothendieck's classical group-theoretic legacy to a new "geometrification" of Arithmetic Geometry.* For example, the workshop showed

– the *application of classical approaches* – e.g. formal patching, the "realizing-lifting-parametrizing" program for covers, a section conjecture in localisation – *beyond their original geometric frontier*;

– constructions in *anabelian geometries with an essential abelian nature* – e.g. abelian-by-central extensions, anabelian-motivic use of étale types;

– the *intermingling of analytic and étale Tannaka symmetries* – e.g. the use of automorphic forms in étale local systems;

– the *(re)construction of new arithmetic-geometry contexts* – e.g. the recent developments of anabelian geometry towards understanding $\widehat{GT}$ and $\mathrm{G}_{\mathbb{Q}}$.

The presentations. The workshop developed in three *movements*[1] that, in order to set the tone, we formulate as follows: the first movement in *the arithmetic of (mostly) finite covers*, the second movement of *motivic aspects of Galois representations*, and the third movement with *Galois theory of arithmetic fundamental groups* including in particular aspects of *anabelian geometry*.

The talk schedule started with KAREMAKER's presentation on the towers of iterated Belyĭ maps and their Galois groups in terms of arboreal representations. BARY-SOROKER addressed the topic of the distribution of Galois groups of *random polynomials* starting from the original contribution of van der Waerden to the latest results and some new heuristics.

A few talks focused on complementary aspects of the *realizing-lifting-parametrizing program for Galois covers*, which all originated in the Hilbert's Irreducibility Theorem (HIT). LEGRAND explained to what extent, for a given finite Galois cover f of $\mathbb{P}^1_{\mathbb{Q}}$ with group G, "almost all" Galois extensions of $\mathbb{Q}$ with group G do not occur as specializations of f (in connection to the *abc-conjecture* and *Malle's conjecture*). DÈBES discussed a version "over the ring" of HIT that relates to a polynomial analog of the Schinzel hypothesis. KÖNIG explained how, together with NEFTIN, he could handle the decomposable case of the Hilbert-Siegel problem on the exceptional specialization set. NEFTIN discussed new arithmetic dimensions for finite groups, which measure the existence of Galois parametrizing field extensions of small transcendence degree, and which he compared with the essential dimension. FEHM reported on *the minimal ramification problem* for a finite group G, which asks for the minimal number of ramified primes in a G-Galois extension. He mostly considered the special situation the base field is $\mathbb{F}_q(t)$ and G is S_n or A_n.

FRIED explained how he could refine the original *Modular Tower program* by introducing ℓ-Frattini lattice quotients. This allows, starting from modular curves, to capture the full relation between the regular inverse Galois problem and precisely generalizing Serre's Open Image Theorem.

A. HOSHI gave a survey of recent developments in the *rationality problem for fields of invariants*, e.g. Noether's problem, rationality problem for algebraic tori, rationality problem for quasi-monomial actions; he notably showed negative results by using such birational invariants as flabby class and unramified Brauer group, but also some coming from 2 and 3 dimensional group cohomology. HARBATER showed how *patching methods*, originally designed for inverse Galois theory, can also be used for direct Galois theory; a main example he developed was about the local-global principle: if a G-torsor is locally trivial, must it be trivial?

Several talks dealt with *motivic aspects of Galois representations*, which are at once of algebraic and analytic in nature – see the (algebraic cycles) Tate and (automorphic forms) Langlands conjectures respectively. The realization of exceptional Lie groups (YUN, PATRIKIS et al.) and a back-and-forth from Galois

[1]The original branches are now so intermingled that the word "movement" seems appropriate. We also allude to the remark of an eminent researcher in the field who spoke of the *MFO symphony* in describing the magical atmosphere of in-person workshops at MFO.

to Tannaka (Collas, Dettweiler et al., see MFO18 report), sketches some bridges, but interconnections run *structurally deeper*. A coherent net of conjectures provides indeed a multi-faceted insight of the motivic properties of ℓ-adic representations: in terms of deformation of local systems (via Simpson's rigidity-integrality of motivic nature, Patrikis et al.), of the isotypicity of cusp forms for a Lefschetz-like trace formula (via global Langlands, Yu), or of family of Picard groups (via Tate-conjecture for K3 surfaces, Tang et al.). We refer to Cadoret's work (joint with Tamagawa) on the theory of local systems with ultra-product and (over)convergent coefficients that illustrates further the interplay of these Tannaka, Frobenius weight, Langlands/companions theories. Collas displayed a blackboard from étale topological type to Morel-Voevodsky motivic homotopy type for schemes and stacks, emphasizing how the Quillen formalism connects functorially the anabelian and cohomological contexts in terms of Grothendieck's conjectures.

New *anabelian geometry with minimalistic or close-to-abelian* insights were explained in several talks. Lüdkte reported on minimalistic anabelian geometry for localised p-adic curves, a result in between the birational case and the case of p-adic algebraic curves. Topaz explained (joint work with Pop) how a two-step version for complements of line arrangements on the plane allows to complete Pop's pro-ℓ abelian-by-central variant of the Ihara/Oda-Matsumoto problem leading to a new "linear Grothendieck-Teichmüller group" and thus a new $G_{\mathbb{Q}}$-characterization. Pop presented work (joint with Topaz) on a minimalistic Saïdi-Tamagawa's m-step Neukirch-Uchida theorem.

In a reverse direction, Horel discussed a non-trivial action of $\widehat{\mathrm{GT}}$ on the configuration category of $\mathbb{R}^d$ ($d \geq 2$) and reported an application to computing the higher homotopy groups of the space of knots in $\mathbb{R}^d$ localized at a prime via the Goodwillie-Weiss manifold calculus.

A series of talks presented more variants of the latest results in anabelian geometry of curves and configuration spaces: Y. Hoshi reported a new class of geometric objects, the quasi-tripods, that provides relative and absolute anabelian results in higher dimensions. Sawada discussed how the graded Lie algebra structure associated to a fundamental group recovers the geometric type of configuration spaces of a hyperbolic curve. Minamide reported a result determining the outer automorphism group of a profinite braid group in terms of the profinite Grothendieck-Teichmüller group $\widehat{\mathrm{GT}}$.

Balakrishnan reported on recent progress on the quadratic Coleman-Chabauty-Kim p-adic integration method for Diophantine problems and for new applications to the determination of rational points on modular curves. The question of single valued constructions in terms of canonical path in the archimedean and p-adic and tropical settings of iterated integration was presented by Litt – see also Litt's MFO18 report. Pries presented formulas that describe the Galois actions on mod p central series of the fundamental group of a Fermat curve in view of Anderson's work. She illustrated their consequences with computational

applications and open questions. Considering the Anderson-Ihara-Wojtkowiak ℓ-adic Galois associator, where Coleman-type iterated integral theory is unavailable, NAKAMURA discussed functional equations between ℓ-adic Galois multiple polylogarithmic functions on the absolute Galois group.

Poster session for Oberwolfach Leibniz Fellows. In addition to the oral communications above, *a poster session* via the Slack workspace was organized for the OWLG fellows to introduce their field of research: (1) PHILIP uses ℓ-adic monodromy techniques to study the *semi-stability degree of abelian varieties* over number fields; (2) SHIRAISHI develops the ℓ-adic *Galois polylogarithms for a reciprocity law* of the triple ℓth power residue symbols; and (3) YUJI deals with the *categorical reconstruction of a scheme* S being given its abstract category of S-schemes Sch_S.

A workshops that never sleeps. Over the week, the online workshop gathered 37 participants distributed over 3 main time-zones (US: 9, EU: 18, JP: 10). The program consisted of 26 forty-minutes presentations each one followed by a twenty-minute extension time for informal discussion with the audience. Most of the talks were given live on `Zoom`, at least one was pre-recorded, and due to some technical problem, one was even a one-blackboard presentation.

The recordings of the talks were paired with a dedicated `HGGT-MFO21 slack workspace` for *asynchronous* comments and questions, and with 2 three-hours daily "gather sessions" on `gather.town` for *live encounters* of all the participants.

Thanks to the commitment of the participants (with an average of 21 ± 3 participants per talk, full 20-minute discussions for all talks, and $\sim 1\,500$ slack messages (25% private) over 5 days), it resulted in some lively stimulating exchanges, some fierce discussions, and *a HGGT workshop that never sleeps.*

This workshop confirmed the momentum initiated in the previous MFO18 mini-workshop and in the Tatihou meeting on *Field Arithmetic and Arithmetic Geometry* ("Rencontres arithmétiques de Caen 2019"). Following the strong support and feedback of the participants, agreement has been made to meet again within the next two years for reporting on the new research lines that appeared during this workshop.

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Workshop (online meeting): Homotopic and Geometric Galois Theory

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