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Mini-Workshop: Arithmetic Geometry and Symmetries around Galois and Fundamental Groups

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Abstract. The geometric study of the absolute Galois group of the rational numbers has been a highly active research topic since the first milestones: Hilbert's Irreducibility Theorem, Noether's program, Riemann's Existence Theorem. It gained special interest in the last decades with Grothendieck's "Esquisse d'un programme", his "Letter to Faltings" and Fried's introduction of Hurwitz spaces. It grew on and thrived on a wide range of areas, *e.g.* formal algebraic geometry, Diophantine geometry, group theory. The recent years have seen the development and integration in algebraic geometry and Galois theory of new advanced techniques from algebraic stacks, ℓ-adic representations and homotopy theories. It was the goal of this mini-workshop, to bring together an international panel of young and senior experts to draw bridges towards these fields of research and to incorporate new methods, techniques and structures in the development of geometric Galois theory.

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Introduction by the Organisers

The workshop *Arithmetic Geometry and Symmetries around Galois and Fundamental Groups* dealt with recent progress in the study of the absolute Galois group of the rational numbers based on geometric representations through étale fundamental groups. This includes various approaches which aim at moduli properties of algebraic spaces *via* their arithmetic-geometric interpretations, and which translate into the study of the finite quotients of the absolute Galois group – the *Inverse Galois Problem.* IGP has been one key thread of the workshop, a second one being the *symmetries of the spaces*, which the automorphisms of the structures reflect.

Overview

The traditional *Geometric Galois & Inverse Galois* approaches – abelian, geometric, *via* Galois representation – have grown into new branches leading to striking results: conditions on Galois realizations expressed in terms of rational properties of Hurwitz space towers (Fried, Dèbes, Cadoret, Tamagawa), arithmetic properties of the stack structures of moduli spaces of curves (Schneps, Nakamura, Collas, Maugeais), extension of anabelian results to higher-dimension (Hoshi, Schmidt, Stix), realization of new groups as Galois groups (Dettweiler, Reiter), specialization properties of geometric Galois realizations (Dèbes, Legrand, König, Neftin), contributions to Colliot-Thélène's program on G-torsors (Harari, Wittenberg).

The workshop focused on this progress as organized under 3 *hot topics*:

(1) *Abelian approach to Inverse Galois.* After the completion of the Shafarevich solution to IGP for solvable groups, the Colliot-Thélène approach to the Noether program and the Grunwald problem, *via* the study of rational points on rationally connected varieties, has become a leading project.
(2) *Geometric Galois Theory*, which investigates the arithmetic of finite Galois covers of the projective line and their specializations and has led to the study of their moduli spaces and their towers — Hurwitz spaces and Modular Towers, has been the only key to the non-solvable territory.
(3) *Galois Anabelian and Homotopical Geometry*, which deals with Galois properties of the étale fundamental group as supported by the seminal example of the moduli stack of curves, has been, since Grothendieck, one of the most influential set of ideas.

One goal of the workshop has been to take full advantage of the bridges between the three topics, notably by considering the *geometric and arithmetic higher symmetries* of the objects *via* homotopical methods. The automorphisms of families at the 2-categorical level of Hurwitz and moduli stacks, and the higher cohomological obstructions to rationality are two successful examples of this approach. The introductory talk referred to this quote by S. Lefschetz *"It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry"*. Taking over this mission for Arithmetic was inspirational for the workshop.

The program of the mini-workshop consisted of 18 eighty-minutes lectures. Participants introduced their knowledge and shared their progress in a lively atmosphere of stimulating exchanges. Informal sessions crystallizing the connections between classical problems and some of the new expertises revealed quite promising. For example the embedding problem paired with the spectral étale Brauer-Manin obstruction, Regular Inverse Galois Theory with approximation properties on the Noether variety, anabelian geometric properties with étale homotopy type.

This mini-workshop renewed the long and strong tradition of fruitful exchanges between Arithmetic and Galois theories in such famous places as Oberwolfach, Luminy, Seattle, Red Lodge, and Kyoto. Because of the strong support and feedback of the participants, agreement has been made to meet again in a similar event within the next two years for sharing the progress of the research directions initiated during this mini-workshop.

1. Abelian Inverse Galois Theory

The three talks on this topic related to the Brauer-Manin obstruction to existence or density of rational points on a variety over a number field. A classical conjecture due to Colliot-Thélène – the BM obstruction is the only one for geometrically rationally connected varieties (*e.g.* unirational) – link this issue to Inverse Galois Theory. Indeed it is well-known that this conjecture, applied to the Noether variety $\mathbb{A}^d/G$ (with $d = |G|$), or its variant SL_m/G (for some embedding $G \hookrightarrow \mathrm{SL}_m$), leads to a solution of IGP for the group G in question. Furthermore this approach generally also provides some answers to the local-global Grunwald problem – lift a finite set of local Galois extensions of group embedded in G to some global Galois extension of group G. Recently new ingredients coming from homotopy theory, the notion of spectras, have appeared in this topic.

In the first part of his talk, Harari reviewed the classical material around the Brauer-Manin obstruction (over number fields). Then he explained that a big part extends to global fields of positive characteristic, *e.g.* the function field K of a curve over a finite field, or even over more complicated fields, provided some good arithmetic duality properties remain. In this context, he finally presented some recent results obtained (some jointly with Szamuely and some by Izquierdo) for some K-algebraic groups.

Schlank reported on how modern homotopy theory gives – in terms of étale topological type, spaces, and spectras – a potential finer context where to express fundamental arithmetic questions. He first discussed the problems of Grothendieck section conjecture, of Skorobogatov's étale-BM obstruction to rational points, and of Galois embeddings (jw. Carlson) in terms of cohomological methods. He then presented recent developments of this approach within the stable motivic homotopy theory, which reveal a higher cohomological obstruction (jw. Stojanoska).

Wittenberg, after recalling the Colliot-Thélène conjecture and its connection to Inverse Galois Theory, presented his recent joint result with Harpaz: a proof of the conjecture when the variety is a smooth proper model of an homogeneous space V of SL_m with finite and supersolvable stabilizers. In the case $V = \mathrm{SL}_m/G$ with G a finite group, embedded in some SL_m, they obtain as a corollary that every supersolvable group G is a Galois group over any given number field, and even that every Grunwald problem for G, for places not dividing $|G|$, has a solution. He also explained the strategy of the proof and its main ingredients.

2. Geometric Galois Theory

The most basic *Inverse Galois Problem* version is to show that $G_{\mathbb{Q}}$ has every finite group as a quotient. Some significant success has, however, come through the *Regular Inverse Galois Problem* (RIGP) for which the basic tools are sophisticated versions of *Riemann's existence theorem* followed by specialization (Hilbert's irreducibility theorem). The regular approach is driven by the moduli of covers of the projective line $\mathbb{P}^1$ – Hurwitz spaces. The profinite nature of Galois groups leads to their organization in towers – Modular Towers, which also takes us back to fundamental ℓ-adic representation issues. Recently there has also been a focus on the specialization process itself aimed at assessing the difference between the two inverse Galois problems.

Fried's talk applied his **M**(odular)**T**(ower) generalization of modular curve towers to the **R**(egular)**I**(nverse)**G**(alois)**P**(roblem) and expanding Serre's **O**(pen) **I**(mage)**T**(heorem). From any finite ℓ-perfect (ℓ prime) group G, a characteristic extension, $V_\ell \stackrel{\text{def}}{=} (\mathbb{Z}_\ell)^{\nu(G,\ell)} \to \tilde{G}_\ell \to G$, leads to towers of Hurwitz spaces based on the finite group quotients $\tilde{G}_\ell/\ell^{k+1}V_\ell \stackrel{\text{def}}{=} {}_\ell^k G$, $k \geq 0$. An example used his formula for computing all expected properties – genus, cusps, degree, fine moduli properties – of j-line covers by reduced Hurwitz spaces of 4-branch point covers. His concluding examples showed **MT**s to be a *seam* between the **OIT** and the **RIGP** enhancing Fried's two main **OIT** conjectures.

Dèbes followed Raynaud's "one-slide" tradition to present a diagram showing the state of the art in Inverse Galois Theory and structuring it in three categories of problems: realizing, lifting, parametrizing. He then used the same diagram to recast a series of recent results from a joint program with Koenig, Legrand and Neftin on the specialization process. In various situations, they show that the sets of Galois extensions obtained by specialization from natural sets of Galois covers of the line of fixed group G (singletons, moduli spaces) are big (in some density sense), but also cannot be too big (*e.g.* they generally do not contain all Galois extensions of group G). More detailed talks by his co-authors were to follow.

Legrand presented in more details some of the specialization results mentioned in Dèbes' talk. He emphasized his results on the parametricity property. No group G had failed having a parametric extension over a given number field k: a regular Galois extension $F/k(T)$ that parametrizes, *via* specialization of T in $\mathbb{P}^1(k)$, all Galois extensions E/k of group $G = \mathrm{Gal}(F/k(T))$. A joint work of König and his now offers many such groups: abelian $\neq C_4, C_p$, S_n ($n \geq 4$), *etc.* He also discussed analogous results for the *regular* type of specialization, for which T is specialized in $k(U)$, and so the outcome is really a rational pull-back cover.

König came back to the specialization approach to the Grunwald problem alluded to in Dèbes's talk, which consists in using the specialized extensions of some regular realization of a group G to solve Grunwald problems for this group. After recalling the unramified case (after Dèbes-Ghazi), he showed, based on a joint result of Legrand, Neftin and his on decomposition groups of specialized extensions, that the Grunwad problem cannot be handled by the specialization approach in general, but that promising perspectives exist if a 1-parameter family of regular realizations of G is available. He also explained how to use their local work to produce new groups with no parametric extensions, *e.g.* A_n ($n \geq 4$).

Neftin focused on an old famous problem related to Hilbert's Irreducibility Theorem, which is to investigate situations for which the exceptional "reducible set" in Hilbert's theorem is finite. A breakthrough in this problem, due to Fried, was to understand how it is governed by group theory, *via* the monodromy groups of the associated covers. Neftin recalled the special problem where the initial polynomial is of the form $p(Y)-T$ for which the expected conclusions (the reducible set is finite except for the values of p) have been obtained if p is indecomposable and $\deg(p) \neq 5$ (Fried). Neftin explained new results in this context, obtained by Zieve and him, together with some further results, by König and him, on the decomposable case using Ritt's theorems on decompositions of rational functions.

Dettweiler presented a recent joint work with Collas and Reiter on the category of perverse sheaves over elliptic curves which is Tannakian with respect to the convolution product. He showed how this allows some classical Galois realization methods to go *beyond the rational rigidity barrier*. After presenting an alternative to Hilbert's Irreducibility Theorem in terms of a Mordell-Weil rank criterion for local systems, he explained how the convolution approach relies on computations of the monodromy within elliptic braid groups, and provided some examples.

Cadoret reported on a variant with ultraproduct coefficients of the fundamental theorem of *Weil II for curves*. She first recalled Deligne's theory of Frobenius weights for lisse $\overline{\mathbb{Q}}_\ell$-coefficients and its application to the semisimplicity of geometric monodromy. Considering the issue of extending the semisimplicity of geometric monodromy to integral and modulo-ℓ coefficients arising from arbitrary compatible systems of lisse $\overline{\mathbb{Q}}_\ell$-sheaves, she motivated the introduction of a category of "almost-curve tame" ultraproduct coefficients. She explained to what extend this category is well-behaved and why it can be used to develop a theory of Frobenius weights paralleling the one of $\overline{\mathbb{Q}}_\ell$-coefficients. Finally, she elaborated on further applications of her theory such as torsion freeness and unicity of integral models in arbitrary compatible systems of lisse $\overline{\mathbb{Q}}_\ell$-sheaves or the construction of the automorphic to Galois direction of a Langlands correspondence with ultraproduct coefficients.

3. Galois Anabelian and Homotopical Geometry

Broadly, anabelian geometry deals with the arithmetic properties of finite étale covers of a space, which relies on the study of Geometric Galois representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ in the étale fundamental group of spaces. With the goal of identifying arithmetic invariants within topological constructions, it relies on the essential example of moduli stack of curves which provides a computational context – for example via the Grothendieck-Teichmüller theory–, and also a connection to the theory of motives. New fruitful research directions to be pursued appeared during the workshop, which includes simplical and homotopical methods via étale topological type, (unstable) motivic homotopy theory and operads, as well as higher genus and stack considerations.

Nakamura reported on problems and recent progress of the study of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ via the tower of the universal monodromy representations in the moduli spaces of marked curves. After some reminder on Anderson-Ihara Beta function, Soulé and Kummer characters in meta-abelian quotients and their relation in genus 0 to the profinite Grothendieck-Teichmüller group $\widehat{GT}$, he presented his recent work in genus 1 on Eisenstein cocycles and Enriquez' group $\widehat{GT}_{ell}$, and how they are the analogs of the genus 0 constructions.

Schmidt and Stix presented their joint work: they showed how to use étale homotopic methods and Mochizuki's work to deduce the existence of anabelian Zariski-neighbourhoods in smooth variety of any dimension. Schmidt first explained the necessary requirements and difficulties in Artin-Mazur-Friedlander pointed-unpointed étale homotopy theory, then Stix presented the proof based on Tamagawa's idea of Jacobian approximation of rational points via the existence of a certain retract.

Collas presented the divisorial and stack inertia arithmetic contexts of the moduli stacks of unordered marked curves, and their key role in Geometric Galois representations, anabelian geometry, and mixed Tate motivic theory. After discussing his joint work with Maugeais on the Tate-like Galois action on cyclic stack inertia, its connection to Inverse Galois theory via Hurwitz spaces and the fundamental role of Harris/Deligne-Mumford compactification, he presented a work in progress on how the homotopical approach leads to stacky constructions in Morel-Voevodsky's motivic homotopy category, as well as to computable stack periods.

Litt presented his work on étale Geometric Galois representations via the Malčev completion of the pro-ℓ-completion of fundamental group of algebraic varieties. He reported on a joint work with Betts on the semisimplicity of Frobenius actions on ℓ-adic and p-adic (log-crystalline) pro-unipotent fundamental groups, with application to the irreducibility of Kim-Selmer varieties in Chabauty-Kim theory, and results on the representation theory of arithmetic fundamental groups. As an archimedean analogue, he produced and explained the role of canonical paths in the computation of iterated integrals, and used them to recover various special functions of Bloch-Ramakrishnan-Zagier.

Borne discussed some joint works with Biswas and Vistoli on the construction of cyclic ramified covers of curves via the stack of roots and the notion of weighted parabolic sheaves. After a recollection on Mehta-Seshadri's work on weighted parabolic bundle, Nori's fundamental group scheme, and Noohi's automorphisms uniformisation criterion for stacks, he presented his result in terms of Nori uniformization Tannakian criterion.

Quick presented his results on obstructions for the algebraicity of topological cycles via cobordism and simplicial homotopy theory of presheaves. He first showed how the stable motivic homotopy theory for smooth varieties over finite fields allows the construction of cobordism invariants that can be used to detect and construct non-algebraic classes. He also mentioned some arithmetic prospects in Arakelov theory.

Wickelgren reported on a joint work in progress with Westerland on the π_1-sections for configuration spaces. Their approach relies on the use of tangential base points in connection to the parenthesized braid group operad PaB. After presenting how they are linked together, she recalled the Drinfel'd-Fresse definition of $\widehat{GT}$ as homotopy automorphism group of $\widehat{PaB}$. This enriches the monoidal structure on Hurwitz spaces given by juxtaposing conjugacy classes with the intent to algorithmically produce families of special points on Hurwitz spaces.

FIGURE 1. Overview of the Mini-workshop.

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