ON GALOIS ACTION ON STACK INERTIA OF MODULI SPACES OF CURVES

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Abstract. We establish that the Galois action on the cyclic stack inertia of the moduli spaces of curves is given by *cyclotomy conjugacy*. This result follows a similar result for the specific type of inertia without étale factorisation previously established by the authors. It is here extended to the general case by comparing deformations of Galois actions.

The *cyclic* inertia being amongst the most elementary pieces of the stack inertia, this result strengthens the analogy with inertia at infinity, and opens the way to a more systematic Galois study of the stack inertia through the corresponding stratification of the space.

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1 Introduction

In this paper, we give a final answer to the question of characterizing the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on *cyclic stack inertia* of the moduli spaces of marked curves $\mathcal{M}_{g,[m]}$, as initiated in [3, 2, 4] using first Grothendieck-Teichmüller and then étale cohomology theories.

For a given geometric point $x \in \mathcal{M}_{g,[m]} \times \bar{\mathbb{Q}}$, the choice of a $\mathbb{Q}$-point of $\mathcal{M}_{g,[m]}$ defines a geometric representation $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}[\pi_1(\mathcal{M}_{g,[m]} \times \bar{\mathbb{Q}}, x)]$. Following Grothendieck, the study of geometric representation in moduli spaces of curves has been conducted from the point of view of the *Deligne-Mumford stratification* – with results on *divisorial inertia* – then leading to the Grothendieck-Teichmüller theory as developed for example in [17].

Let $I_{\mathcal{M}}$ be the stack inertia of the Deligne-Mumford stack $\mathcal{M}_{g,[m]}$. A geometric $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation then defines a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the set of conjugacy classes of the stack inertia of a point $I_x = I_{\mathcal{M},x} \hookrightarrow \pi_1(\mathcal{M}_{g,[m]} \times \bar{\mathbb{Q}}, x)$ for any geometric point x of $\mathcal{M}_{g,[m]}$ – see §4.1.1 or [4] §2.2. This $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on stack inertia – or group automorphisms of curves – gained some focus only recently: first in [11] with the key role of curves with special automorphisms in Grothendieck-Teichmüller theory in genus 0, then with the study of new $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-relations coming from quotient by finite automorphisms between curves [18, 2], and also with the comparison between $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equations of divisorial type and of stack inertia type [21], [2] §3.

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From a geometric point of view, this stack inertia gives rise to a stratification of the stack $\mathcal{M}_{g,[m]}$ by the irreducible components of the closed substacks $\mathcal{M}_{g,[m]}(G)$ – the so-called *special loci* associated to a finite group G – which is defined as the locus of points $x \in \mathcal{M}_{g,[m]}$ whose stack inertia group I_x contains a subgroup isomorphic to G. Our study takes place in this context of *stack inertia stratification* and is guided by the analogy between its Galois properties and the *Deligne-Mumford stratification*.

1.1 The Stack Inertia Stratification

Following Grothendieck's *Esquisse d'un programme*, the *stratification by topological type* of the Deligne-Mumford compactification of stable curves $\bar{\mathcal{M}}_{g,[m]}$ – which is given by the Knudsen morphisms – led to the development of the arithmetic of the moduli spaces of curves through Grothendieck-Teichmüller theory. This consideration motivates our approach in the study of the *stack inertia stratification* $\mathcal{M}_{g,[m]}$.

1.1.1. As a Deligne-Mumford stack, the moduli $\mathcal{M}_{g,[m]}$ admits an *inertia stratification* given by a finite disjoint union of locally closed substacks – see [10] Theorem 11.5:

$$\mathcal{M}_{g,[m]} = \coprod_{G \in \mathcal{FG}} \mathcal{M}^{in}_{g,[m]}(G),$$

where the $\mathcal{M}^{in}_{g,[m]}(G)$ are gerbes over their coarse moduli space with finite group G as a band, and $\mathcal{FG}$ denotes a set of representatives of the isomorphism classes of the finite subgroups of $\pi_1^{orb}(\mathcal{M}_{g,[m]}(\mathbb{C}))$. The gerbes $\mathcal{M}^{in}_{g,[m]}(G)$ are called the *strict special loci associated to G*

$$\mathcal{M}^{in}_{g,[m]}(G) = \{C \in \mathcal{M}_{g,[m]},\ \mathrm{Aut}(C) \cong G\}$$

and come with stack monomorphisms $\mathcal{M}^{in}_{g,[m]}(G) \hookrightarrow \mathcal{M}_{g,[m]}$. Note that identifying $\pi_1^{orb}(\mathcal{M}_{g,[m]}(\mathbb{C})$ with the mapping class group of surfaces, a result of Nielsen-Kerckhoff ensures that these special loci are non-empty – and the finiteness of $\mathcal{FG}$ follows from the Hurwitz' upper-bound. Consider $\mathcal{FG}$ ordered by inclusion, and let H, $G \in \mathcal{FG}$ such that $G \subsetneq H$. Then the closure of $\mathcal{M}^{in}_{g,[m]}(G)$ is contained in the *special loci* $\mathcal{M}_{g,[m]}(G)$ and satisfies $\mathcal{M}_{g,[m]}(G) \supset \mathcal{M}^{in}_{g,[m]}(H)$. This defines a decreasing dimensional stratification $\{{}_k\mathcal{M}^{in}_{g,[m]}(G)\}_k$, where the strata ${}_0\mathcal{M}^{in}_{g,[m]}(\mathrm{Id})$ of level 0 is given by the stack of curves with trivial automorphisms, and where the cyclic groups appear among the stratas ${}_1\mathcal{M}^{in}_{g,[m]}(G)$ and ${}_2\mathcal{M}^{in}_{g,[m]}(G)$ of level 1 and 2.

In the case of a *cyclic group*, the irreducible components of $\mathcal{M}_{g,[m]}(\mathbb{Z}/n\mathbb{Z})$ admits a combinatorial description given by the *branching data* of the corresponding $\mathbb{Z}/n\mathbb{Z}$-covers – see [1] for p-stratas and n-stratas in $\mathcal{M}_g$, and [4] for n-stratas in $\mathcal{M}_{g,[m]}$ – which is used for the study of the stratas ${}_k\mathcal{M}^{in}_{g,[m]}(\mathbb{Z}/n\mathbb{Z})$.

In the same way that the arithmetic of the Deligne-Mumford stratification of $\bar{\mathcal{M}}_{g,[m]}$ is based on the *irreducible components of the low modular-dimension stratas of this stratification*, the irrreducible components of the stratas of level 1 and 2 may contain already most of the arithmetic of the stack inertia stratification.

1.1.2. The characterization of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on irreducible components of ${}_k\mathcal{M}^{in}_{g,[m]}(G)$ has been initiated as a consequence of [11] within the *first stratas* ${}_1\mathcal{M}^{in}_{0,[4]}(\mathbb{Z}/2\mathbb{Z})$, ${}_1\mathcal{M}^{in}_{0,[4]}(\mathbb{Z}/3\mathbb{Z})$ and ${}_1\mathcal{M}^{in}_{0,[5]}(\mathbb{Z}/5\mathbb{Z})$. It has then been completed for every irreducible stratas $\{{}_1\mathcal{M}^{in}_{g,[m]}(\mathbb{Z}/p\mathbb{Z})\}_{g,m,p}$ as a consequence of the main Theorem 5.4 of [4] by étale cohomology theory, which also gives a partial result for the *second stratas* ${}_2\mathcal{M}^{in}_{g,[m]}(\mathbb{Z}/n\mathbb{Z})$ when the associated $\mathbb{Z}/n\mathbb{Z}$-cover admits no étale factorization – i.e. when the stabilizers of the G-action generate the inertia group. Another partial result for such stratas of level 2 is given for the irreducible component of ${}_2\mathcal{M}^{in}_{g,[m]}(\mathbb{Z}/p^n\mathbb{Z})$ by Grothendieck-Teichmüller theory in the case of genus 0, 1 and 2 – Cf. [3, 2]

The main result of this paper is the extension of this characterisation of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the first and second stratas ${}_1\mathcal{M}^{in}_{g,[m]}(\mathbb{Z}/n\mathbb{Z})$ and ${}_2\mathcal{M}^{in}_{g,[m]}(\mathbb{Z}/n\mathbb{Z})$ to the case of irreducible components associated to *any cyclic stack inertia* – see Theorem 4.8:

Theorem (A). *Let $G = <\gamma>$ be a cyclic stack inertia group of $\mathcal{M}_{g,[m]}$. The Galois action on G is given by χ-conjugacy, i.e. for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$:*

$$\sigma.\gamma = \rho_\sigma\, \gamma^{\chi(\sigma)}\, \rho_\sigma^{-1} \quad \textit{for } \rho_\sigma \in \pi_1(\mathcal{M}_{g,[m]} \times \bar{\mathbb{Q}}),$$

where $\chi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbb{Z}}^*$ denotes the cyclotomic character.

Note that in the case of the Deligne-Mumford stratification and the associated procyclic divisorial inertia, the exact analogous result has been achieved first by a case-by-case study of degeneracy of curves [14, 15, 16] by Grothendieck-Murre theory, then in an exhaustive way in [17] by Grothendieck-Teichmüller theory.

Our approach is to relate *degeneracy of stack inertia stratas* with *Deligne-Mumford stratas* by building G-equivariant deformations of curves, thus allowing the comparison of irreducible component in $\mathcal{M}^{in}_{g,[m]}(G)$ and $\mathcal{M}^{in}_{g-1,[m]+2}(G)$. The $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action naturally relies on the study of tangential base points for Deligne-Mumford stacks whose motivation initially comes from the Deligne-Mumford stratification.

1.2 Inertial Limit Galois Action and G-Deformation

An inertial limit Galois action is the process of studying the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the inertia group of a curve through the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on one of its degeneracy in the boundary of $\mathcal{M}_{g,[m]}$. This process relies on the notion of tangential base point for stack and on Knudsen-compatible G-equivariant deformation of curves.

1.2.1. Subsequently to the original P. Deligne's definition of tangential base point on $\mathbb{P}^1 - \{0, 1, \infty\}$, Y. Ihara and H. Nakamura introduced the notion of tangential base point on $\mathcal{M}_{g,m}$ in [8] based on the theory of formal deformation of curves. It was then understood to be a key construction for the study of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions on curves at infinity in the Deligne-Mumford compactification, as well as for the comparison of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions on different Deligne-Mumford-stratas – see for example [13].

We develop here a generic construction of *rational tangential base point for Deligne-Mumford algebraic stacks*, which naturally takes the stack structure into account. Initiated by [23] for geometric points, we extend it to k-rational base points in order to obtain tangential $\mathrm{Gal}(\bar{k}/k)$-actions on Deligne Mumford algebraic k-stacks and inertia groups, with the following result – see Proposition 3.6 for general statement, Corollary 3.12 for curves and Lemma 4.4 for inertia groups:

Theorem (B)**.** *Let $\mathcal{M}_{g-1,[m]+2}$ denotes the moduli space of stable curves of genus $g-1$ with m marked points and 2 fixed points and $\widetilde{\mathcal{M}}_{g,[m]}$ a partial compactification of $\mathcal{M}_{g,[m]}$. There exist $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions on $\pi_1(\mathcal{M}_{g-1,[m]+2}\times\bar{\mathbb{Q}})$ and $\pi_1(\mathcal{M}_{g,[m]}\times\bar{\mathbb{Q}})$ with a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant morphism*

$$\pi_1(\mathcal{M}_{g-1,[m]+2}\times\bar{\mathbb{Q}})\longrightarrow\pi_1(\widetilde{\mathcal{M}}_{g,[m]}\times\bar{\mathbb{Q}})$$

which sends the cyclic stack intertia to the cyclic stack inertia.

Our approach follows the Grothendieck-Murre formalism of tame fundamental group $\pi_1^{\mathcal{D}}(\mathfrak{M})$ associated to normal crossing divisor $\mathcal{D}$ on a stack $\mathfrak{M}$, and is strongly inspired by the original definition of a tangential base point as canonical generators of a formal neighbourhood of a point as given in [8]. This theory of $\mathbb{Q}$-*tangential base points for Deligne-Mumford stacks*, developed in §3.2, leads to a commutative diagram, with β_I defined *on the cyclic groups inertia $I_{\mathcal{M},x}$ only*:

$$\begin{array}{ccc}\pi_1(\mathcal{M}_{g-1,[m]+2}\times\bar{\mathbb{Q}}) & \overset{\beta_I}{\dashrightarrow} & \pi_1(\widetilde{\mathcal{M}}_{g,[m]}\times\bar{\mathbb{Q}})\\ \Vert & & \downarrow\\ \pi_1^{\mathcal{D}^*}(\bar{\mathcal{M}}_{g-1,[m]+2}\times\bar{\mathbb{Q}},\vec{s}) & \xrightarrow{\beta_\diamond} & \pi_1^{\mathcal{D}}(\bar{\mathcal{M}}_{g,[m]}\times\bar{\mathbb{Q}},\vec{s}\,')\end{array}$$

where $\beta_\diamond$ denotes the clutching morphism of Knudsen, $\widetilde{\mathcal{M}}_{g,[m]}$ a partial compactification of $\mathcal{M}_{g,[m]}$, and $\vec{s}$ and $\vec{s}\,'$ are compatible tangential base points – note that the equalities are actually defined through the choice of the base points. An *inertial limit Galois action* on an inertia group $\widetilde{I}<\pi_1(\mathcal{M}_{g-1,[m]+2}\times\bar{\mathbb{Q}})$ is then a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action which is induced by a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on a cyclic inertia group $I<\pi_1(\mathcal{M}_{g,[m]}\times\bar{\mathbb{Q}})$ deduced from the Knudsen morphism β and the choices given in Theorem (B).

In the same way that the *limit Galois action on curves* of Y. Ihara and H. Nakamura led to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on divisorial inertia groups of different Deligne-Mumford stratas of $\mathcal{M}_{g,[m]}$, this *inertial limit Galois action* with respect to the inertia stratification of §1.1 is crucial in the comparison of the arithmetic of the different stratas $\{\mathcal{M}_{g,[m]}^{in}(G)\}_G$.

This inertial limit Galois action allows us in §4.2.3 to reduce the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-inertial action to stratas of lower modular dimensions through β_I – which strengthens the analogy with the divisorial inertia, for which this approach is a key tool in the study of its arithmetic – it nevertheless relies on the control of specific degeneracies of curves. This behaviour also gives a motivation to lead future and more detailed comparisons between the two stratifications, through a better understanding of the degeneracies.

1.2.2. The study of the morphism β_I above at the level of cyclic stack inertia group is led from the point of view of degeneracy of irreducible components of the special locus $\mathcal{M}_{g,[m]}(G)$ in $\mathcal{M}_{g,[m]}$. More precisely, a fixed conjugacy class of a cyclic group G corresponding to an irreducible component of the special locus as described in [4], the question is to characterise degenerated curves in the boundary of the partial compactification $\widetilde{\mathcal{M}}_{g,[m]}$ relatively to the Knudsen morphism β – see §3.2.1. This is obtained by G-equivariant deformation theory in Corollary 2.7 and in §4.2.2 at the level of inertia groups:

Theorem (C)**.** *For G a cyclic inertia group of $\mathcal{M}_{g,[m]}$, any generic point of the special loci $\mathcal{M}_{g,[m]}(G)$ is obtained as a generisation of a curve in $\bar{\mathcal{M}}_{g,[m]+2}$ which has no étale factorisation. Moreover, the property of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action to be given on stack inertia groups by χ-conjugacy is stable under specialisation.*

This result relies on the ability of recovering the global Hurwitz datas of a cyclic G-action by the local reading of the G-action on the tangent space of a ramification point – the so-called γ-type, see Definition 2.3. We hence obtain an equivalent notion of branching datas for singular curves so we can link them to the datas of their normalisation and keep track of them during a deformation.

Let us now summarise how the main Theorem (A) relies on Theorems (B) and (C): for any given cyclic automorphism group G of $\mathcal{M}_{g,[m]}$ – that corresponds to a strata $\mathcal{M}_{g,[m]}(G)$ – the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on G first reduces to the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on an irreducible strata $\mathcal{M}_{g-1,[m]+2}(G)$ by Theorem (B). Since we can suppose it is without étale factorisation by Theorem (C), the Galois χ-action on G then results from Corollary 4.6, which is similar to a construction in [4].

1.3 Towards Anabelian Considerations

In the field of Anabelian Geometry – i.e. the study of properties which allow to recover the isomorphism class of a variety X/k from its Galois outer representation $\mathrm{Gal}(\bar{k}/k) \to \mathrm{Out}[\pi_1(X \times \bar{k})]$ – it has been a key ingredient of giving *group theoretic characterisations of arithmetic properties*, as in the case of divisorial inertia groups, to obtain anabelian properties of curves. From this point of view, Theorem (A) naturally raises the question:

Is the Galois χ-action on the cyclic stack inertia of moduli spaces of curves an anabelian property of the Deligne-Mumford stack $\mathcal{M}_{g,[m]}$?

The profinite nature of the geometric fundamental group also motivates a profinite and more precise version of the question above – for comparison with the divisorial formulation, we refer to [12] Theorem 3.4 and [22] §7.C:

When the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action is given by cyclotomy on a protorsion element of $\pi_1(\mathcal{M}_{g,[m]} \times \bar{\mathbb{Q}})$, is this element conjugate to a finite stack inertia element?

Let us identify $\pi_1(\mathcal{M}_{g,[m]} \times \bar{\mathbb{Q}})$ with the profinite completion $\widehat{\Gamma}_{g,[m]}$ of the mapping class group – or group of diffeomorphims of surfaces. The group $\Gamma_{g,[m]}$ being residually finite, the stack inertia is part of the protorsion of $\widehat{\Gamma}_{g,[m]}$. Theorem (A) then gives a necessary condition for the question above on the subclass of the stack inertia in $\Gamma_{g,[m]}$ inside the profinite torsion in $\widehat{\Gamma}_{g,[m]}$. Note that at the level of profinite torsion, only little is known, and we refer to [3, 2] and [4] for analogous results in the case of gens $0, 1$ and 2 for p-profinite torsion *only*.

2 Geometry of the Special Loci at Infinity

Let G denote a finite group and $\mathcal{M}_{g,[m]}(G)$ the special loci associated to G, i.e. the substack of $\mathcal{M}_{g,[m]}$ composed of curve admitting a subgroup of automorphism isomorphic to G. For curves in $\mathcal{M}_{g,[m]}(G)$, we study a specific type of degeneracy to the boundary of $\mathcal{M}_{g,[m]}$ when G is cyclic. Recall that a smooth G-curve is said to be *without étale factorisation* if the associated G-cover does not factorise through a non-trivial étale cover. We show how any generic G-curve admits a degeneracy to a singular curve which admits a G-action and whose normalisation is *without étale factorisation*.

2.1 Deformation of stable G-Curves

For curves endowed with a G-action, the analogous of the stable curves of the Deligne-Mumford compactification is given by *stable G-curves*, i.e. stable curves endowed with an *admissible action* whose definition is recalled below in the case of a *cyclic group* – we refer to [5] for the general case.

Definition 2.1 (Admissible action)**.** *Let G be a cyclic group acting faithfully on a semi-stable curve $\mathcal{C}/S$. For S the spectrum of an algebraically closed field, the action of G is said to be admissible if, for every singular point $P \in \mathcal{C}$ with stabilizer G_P, the two characters of G_P on the branches at P are each other's inverse. For S general, the action is admissible if it is so on every geometric fibre.*

The Schlessinger's theory of the deformation functor associated to a stable G-curve then establishes that there is no obstruction to the lifting of infinitesimal deformation, i.e. the functor is pro-representable by a complete local ring $R_{C,G}$.

Theorem 2.2 ([5] – Prop. 2.1–2.2)**.** *Let C be a stable G-curve over a field k of characteristic 0 endowed with a G-admissible action at each singular point, and let $R_{C,G}$ be its universal deformation ring with residue field k and field of fraction K. Then $R_{C,G}$ is formally smooth over k and the deformation at the generic point is formally smooth over K.*

More precisely, the cohomological theory controlling the G-equivariant deformation functor of a curve C is given by the k-vector spaces $\mathrm{Ext}^i_G(\Omega_C, \mathcal{O}_C)$, and leads to a *local-global principle* for tame G-covering:

$$0 \to \mathrm{H}^1(C/G, \mathrm{Ext}^0_G(\Omega_C, \mathcal{O}_C)) \to \mathrm{Ext}^1_G(\Omega_C, \mathcal{O}_C) \to \bigoplus_{Q_i \in db(C)} \mathrm{Ext}^1_G(\Omega_{C,Q_i}, \mathcal{O}_{C,Q_i}) \to 0$$

where the direct sum is taken over singular points and branch points of C/G. In particular, the "local" contributions are given by the deformation of the branch and singular points Q_i of C/G, each of them being of dimension 1. The universal global deformation ring identifies to

$$R_{C,G} \simeq R_{glo}\widehat{\otimes}k[[q_1,\dots,q_\ell]].$$

where R_{glo} is a formally smooth k-algebra of finite dimension and ℓ the number of singular and branch points.

2.2 The Case of G-curves without étale factorisation

In the following, we denote by G a cyclic group of order $n \in \mathbb{N}$ and by γ a generator of G. For $\mathbf{kr}$ a given branching data, our degeneration result relies on the abilities of building a stable marked G-curve C/k with branching data related to $\mathbf{kr}$ by gluing in an admissible way points fixed by G, and controlling the branching data through G-equivariant deformation. This is achieved using the notion of γ-type first applied to the case of unmarked, then to marked curves.

2.2.1. For C/k a G-curve with k a field containing n-th roots of unity, recall that locally for the étale topology the G-cover $C \to C/G$ is given by an equation of the form

$$y^n = \prod_I (x-\alpha_i)^{k_i}$$

and in particular, the order of the stabilizer group of α_i is given by $n/\gcd(n,k_i)$. We denote by $\mathbf{k} = \{k_1,\dots,k_\nu\} \in (\mathbb{Z}/n\mathbb{Z})^\nu$ these Hurwitz data associated to the G-cover $C \to C/G$. We introduce the *γ-type* of a point of C, which allows to recover the Hurwitz data from local informations.

Definition 2.3 (γ-type)**.** *Let k be an algebraically closed field of characteristic 0, let G be a cyclic group of order n, $\gamma \in G$ a generator and $\zeta \in k$ a fixed primitive n-th root of unity. Let C/k be a complete smooth curve endowed with a G-action and $P \in C$ be a closed point with non-trivial stabiliser under the action of G.*

The point P is said to be of γ-type ζ if, for a uniformising parameter u of C at P we have

$$\gamma^\ell(u) = \zeta^\ell u \mod u^2 \quad \textit{for all } \ell \in \mathbb{Z} \textit{ such that } \gamma^\ell(P) = P.$$

The γ-type of a point P is denoted by $\mathrm{type}_\gamma(P)$.

Remark that the γ-type of a point is independent of the choice of the uniformising parameter. The following lemma links the local γ-type with the Hurwitz data of the cover $C \to C/G$, and is used in the next section to built a stable G-curve by gluing two points of *inverse γ-types.*

Lemma 2.4. *Let C/k be a complete smooth curve endowed with a G-action, and denote by $\{P_i\}_I$ the ramification points of $C \to C/G$ with $\{k_i\}_I$ as Hurwitz data. Then there exists $\zeta \in k$ such that for all $i \in I$*

$$\mathrm{type}_\gamma(P_i) = \zeta^{j_i \frac{n}{\mathrm{ord}\,(k_i)}}$$

where j_i is the inverse of $k_i \frac{\mathrm{ord}\,(k_i)}{n}$ modulo $\mathrm{ord}\,(k_i)$.

Note that for $a \in \mathbb{Z}/n\mathbb{Z}$, the element $a\frac{\mathrm{ord}\,(a)}{n}$ is well defined in $\mathbb{Z}/\mathrm{ord}\,(a)\mathbb{Z}$.

Proof. By Kummer theory, the morphism $\pi\colon C \to C/G$ is given over the étale locus by an equation of the form $y^n = f(x)$, the action of G being given by $\gamma(y) = \zeta y$. Let w be a uniformising parameter at $\pi(P_i)$ in C/G so that $y^n = w^{k_i} t$ – up to a n-th power, t being an invertible element.

Writing a decomposition $an + k_i j_i = \frac{n}{\operatorname{ord} k_i}$, the element

$$u = y^{j_i \frac{n}{\operatorname{ord} k_i}} w^a$$

is a uniformising parameter of C at P_i and we have

$$\begin{aligned} \gamma(u) &= \gamma(y)^{j_i \frac{n}{\operatorname{ord} k_i}} w^a \\ &= \zeta^{j_i \frac{n}{\operatorname{ord} k_i}} y\, w^a u \\ \gamma(u) &= \zeta^{j_i \frac{n}{\operatorname{ord} k_i}} u, \end{aligned}$$

hence the corresponding γ-type of P_i. □

Remark 2.5. In particular, once a generator γ of G and a primitive n-th root ζ are fixed, the Hurwitz data is read locally through the action of G on the tangent space of a ramification point – thus giving the local contribution of the global Hurwitz data.

2.2.2. We now establish the result of G-equivariant degeneration to the boundary of $\mathcal{M}_g$ – first in the unmarked case, then adapted to the marked one.

Theorem 2.6. *Let $g, g' \geq 1$ integers, $G = \mathbb{Z}/n\mathbb{Z}$, $\mathbf{k} = (k_1, \ldots, k_\nu) \in (\mathbb{Z}/n\mathbb{Z})^\nu$ satisfying*

$$2g - 2 = n(2g' - 2) + \sum_i (\operatorname{ord}(k_i) - 1)\frac{n}{\operatorname{ord}(k_i)} \tag{2.1}$$

$$\sum_i k_i = 0. \tag{2.2}$$

For all $\ell \in (\mathbb{Z}/n\mathbb{Z})^*$ there exists a singular curve C_ℓ/k of genus g endowed with a G-admissible action, with C_ℓ/G of genus g', and such that:

1. *the normalisation of C_ℓ is of genus $g-1$ and has $(k_1, \ldots, k_\nu, \ell, -\ell) \in (\mathbb{Z}/n\mathbb{Z})^{\nu+2}$ for Hurwitz data;*
2. *the generic G-equivariant deformation of C_ℓ is smooth and has $\mathbf{k}$ for Hurwitz data.*

The construction below also illustrates the control of the Hurwitz data along G-equivariant deformations.

Proof. Let E_0/k be a smooth curve of genus $g' - 1$ over an algebraically closed field k, there exists a G-equivariant cover $E_1 \to E_0$ with Hurwitz data $(\mathbf{k}, \ell, -\ell) \in (\mathbb{Z}/n\mathbb{Z})^{\nu+2}$ – this results from the Leray exact sequence in étale cohomology for Galois G-cover of family of curves, see for example exact sequence (3.3) of [4] – and

E_1 is of genus $g-1$ as

$$n(2(g'-1)-2)+\sum_i(\mathrm{ord}\,(k_i)-1)\frac{n}{\mathrm{ord}\,(k_i)}+2(n-1)=2g-4$$
$$=2(g-1)-2.$$

Let $\{P_1,\dots,P_\nu\}$ denote the ramification points with Hurwitz data $\{k_1,\dots,k_\nu\}$ and $\{P_1',P_2'\}$ the points with data $\{\ell,-\ell\}$.

Following lemma 2.4, there exists ζ such that the γ-type of P_i' verifies:

$$\mathrm{type}_\gamma(P_1').\mathrm{type}_\gamma(P_2')=1. \tag{2.3}$$

Let C_ℓ be the curve obtained from E_1 by gluing P_1' and P_2' as in Fig. 1 below. As ℓ is prime to n, the points P_1' and P_2' are both fixed under G so that the curve C_ℓ is endowed with a G-action. Moreover, this action is admissible thanks to (2.3), and it satisfies property 1 of the theorem since E_1 is the normalisation of C_ℓ.

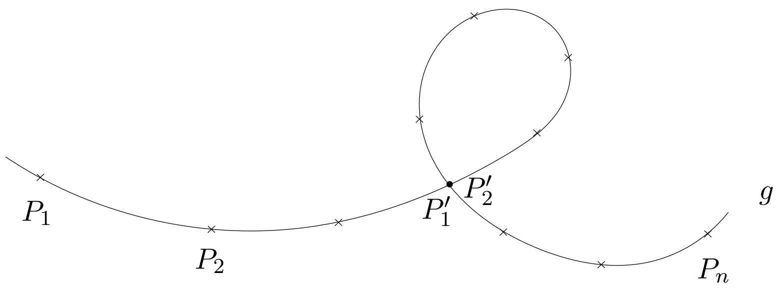


FIGURE 1. Curve C obtained by gluing Q_1 and Q_2.

By theorem 2.2, there exists a G-curve $\mathcal{C}$ over a complete local ring R of residue field k, with special fiber C_ℓ and a generic fiber that is smooth of genus g. Let γ be a generator of G and $\zeta\in k$ a primitive n-th root of unity as fixed by lemma 2.4. For $P_i\in C_\ell$ a ramification point, the action of γ on the local ring $\mathcal{O}_{C_\ell,Pi}\simeq k[[u_i]]$ is given by $\gamma(u_i)=\zeta^{j_i}u_i$ up to a change of parameter u_i. The action of G on $\mathcal{C}$ is then given in a formal neighbourhood of P_i by the same formula on the ring $R[[u_i]]$. Following remark 2.5, the Hurwitz data of the generic fibre of $\mathcal{C}/R$ is then $\mathbf{k}$ as we can read it on u_i through the γ-type so that C_ℓ satisfies property 2. □

Note that the theorem above is still valid under the assumption $\mathbf{k}=\emptyset$. In this case, we obtain a smooth curve with G-action of étale type, which specialise to the curve C_ℓ with only one singular point and whose normalisation is of totally ramified type.

2.2.3. In the case of curves *with m marked points*, i.e. endowed with a horizontal G-equivariant Cartier divisor D of degree m, we recall that the Hurwitz data $\mathbf{k}$ are replaced by the branching data $\mathbf{kr}$– see Definition 3.9 in [4]: the *branching data of a curve $C\in\mathcal{M}_{g,[m]}(G)$* is a couple $\mathbf{kr}=(\mathbf{k},\mathbf{r})$ where $\mathbf{k}$ is a Hurwitz data and $\mathbf{r}=(r_1,\dots,r_n)$ is a n-uple given by:

$$r_i=\#\{y\in D/G,\text{the branching data at } y \text{ is equal to } i\bmod n\}$$

where a n-root of unity is supposed to be fixed. We now state the complete form of our G-deformation result.

Corollary 2.7. *For any generic point $\eta \in \mathcal{M}_{g,[m]}(G)$ there exists a point $z \in \bar{\mathcal{M}}_{g,[m]}(G)$ such that η specialises to z and such that the normalisation of the curve corresponding to z has genus $g-1$ and is without étale ramification.*

Denoting by m' the degree of the divisor D/G, note that in addition to equations (2.1) and (2.2), the datas of the curve corresponding to $\eta \in \mathcal{M}_{g,[m]}(G)$ above are also supposed to satisfy

$$m = \sum_i r_i \gcd(i,n) \tag{2.4}$$

$$m' = \sum_i r_i. \tag{2.5}$$

Proof. This is a direct consequence of the description of the set of the irreducible components of $\bar{\mathcal{M}}_{g,[m]}(G)$ by the set of the branching data **kr** as given in [4], and of Theorem 2.6: for a given **kr**, one constructs explicitly a G-equivariant marked curve as in the proof above. □

We refer to [4] §3.2 (resp. §3.1) for an algebraic definition of **kr** (resp. **k**) for families of curves in term of étale cohomology, and examples. The reference also contains a discussion about the non-canonicity of **k** and **kr** relatively to the choice of a generator γ of G and a primitive n-th root of unity $\zeta \in \mu_n$.

In the final §4.2.3, we precise this G-deformation result at the level of automorphism groups, which is then a key ingredient to reduce the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action by cyclotomy to the case of stack inertia with no étale factorisation.

3 Galois Action at Infinity

We give the definition of Galois actions at infinity attached to a normal crossing divisor of a generic Deligne-Mumford algebraic stack, and then discuss their compatibility through Knudsen morphisms. We clarify this result in the case of $\mathcal{M}_{g,[m]}$, leading to non-canonical comparisons of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations of the fundamental groups of $\mathcal{M}_{g-1,[m]+2}$ and $\mathcal{M}_{g,[m]}$.

3.1 Tangential Base Points

We define a tangential base point for a Deligne-Mumford algebraic stack $\mathfrak{M}$ in term of *the tame fundamental group* of $\mathfrak{M}$ along a normal crossing divisor, which is adapted from the case of scheme in [6]: for a normal crossing divisor $\mathcal{D} \to \mathfrak{M}$, the category $\mathfrak{Rev}^{\mathcal{D}}(\mathfrak{M})$ of tamely ramified covers of $\mathfrak{M}$ along $\mathcal{D}$, is defined via the scheme category $\mathfrak{Rev}^{\mathcal{D}}(X)$ by pull-back along a presentation $X \to \mathfrak{M}$ – see [23] §-3.

3.1.1. Let k be a field and $\mathfrak{M}$ be a Deligne-Mumford algebraic k-stack. A point $x \in \mathfrak{M}(\mathrm{Spec} k)$ is said to have a *Nisnevich neighbourhood* if there exists an étale morphism $f\colon V \to \mathfrak{M}$ with V a scheme, and a point $v \in f^{-1}(x)$ with residue field

k. For a point x having a Nisnevich neighbourhood, we define *the local ring at x in $\mathfrak{M}$*, denoted by $\mathcal{O}^h_{\mathfrak{M},x}$, as

$$\mathcal{O}^h_{\mathfrak{M},x} = \varinjlim_{(v,V)} \mathcal{O}_{V,v}$$

where the limit is taken over the couples (v,V) as above, and $\mathcal{O}^h_{\mathfrak{M},x}$ is the Henselization $\mathcal{O}^h_{V,v}$ for any Nisnevich neighbourhood. Note that for a stack $\mathfrak{M}$ over k and a smooth point x, the completion can be identified to $\widehat{\mathcal{O}^h_{\mathfrak{M},x}} \cong k[[t_1, \dots t_n]]$.

Let $\mathfrak{M}$ be a k-stack, $x \in \mathfrak{M}(\operatorname{Spec} k)$ a smooth point which is supposed to have a Nisnevich neighbourhood and $\mathbf{t} = \{t_1 \dots, t_n\}$ be a system of parameters of $\mathcal{O}^h_{\mathfrak{M},x}$. We define the *Puiseux ring* of $\mathfrak{M}$ at x with respect to $\mathbf{t}$ as the ring

$$\widetilde{\mathcal{O}}^{\mathbf{t}}_{\mathfrak{M},x} = \varinjlim_{\ell} \left(\widehat{\mathcal{O}^h_{\mathfrak{M},x}} \widehat{\otimes}_k k^{sep} \right) [t_1^{1/\ell}, \dots, t_n^{1/\ell}].$$

Remark 3.1. In the case of a *geometric point* x, the local ring $\mathcal{O}^h_{\mathfrak{M},x}$ in the construction of the Puiseux ring above, is replaced by the *strict Henselization* $\mathcal{O}^{sh}_{\mathfrak{M},x}$ which exists without condition – see [10] Rem. 6.2.1.

The following gives a class of points with Nisnevich neighbourhood, which includes the schematic points of $\mathfrak{M}$.

Proposition 3.2. *Let $\mathfrak{M}$ be a Deligne-Mumford k-algebraic stack and $x \in \mathfrak{M}(\operatorname{Spec} k)$. If $\operatorname{Aut}_k(x)$ is a constant group-scheme, then x admits a Nisnevich neighbourhood.*

Proof. Consider the functor $F\colon (k-\text{Art}) \to \text{Ens}$ defined on the artinian k-algebras by the isomorphism classes of objects of $\mathfrak{M}$ which are deformations of points whose images are equal to that of x in $\mathfrak{M}$. As $\operatorname{Aut}_k(x)$ is a constant group-scheme and the diagonal of $\mathfrak{M}$ is unramified, the functor F is actually a sheaf for the étale topology on $(k - \text{Art})$. Theorem 10.10 of [10] then gives an étale presentation with a k-point above x, thus a Nisnevich neighbourhood of x. □

Consider a normal crossing divisor $\mathcal{D}$ on $\mathfrak{M}$ whose support contains x, and let $t_D = \{t_1, \dots, t_n\}$ be a system of parameters of $\mathfrak{M}$ at x such that $\mathcal{D}$ is given in an étale neighbourhood of x by $t_1 \cdots t_m = 0$. A *k-rational tangential base point* on $\mathfrak{M} \setminus \mathcal{D}$ at x is then defined as a fiber functor in term of *Puiseux ring.*

Definition 3.3. *Let $\mathfrak{M}$ be a Deligne-Mumford k-stack, x be a smooth k-point of $\mathfrak{M}$ having a Nisnevich neighbourhood, and $\mathcal{D}$ be a normal crossing divisor on $\mathfrak{M}$ with $t_{\mathcal{D}}$ a system of parameters of $\mathcal{D}$ at x. The k-rational tangential base point associated to $(x, t_{\mathcal{D}})$ is defined as the functor*

$$\begin{array}{rcl} F_x^{t_{\mathcal{D}}}\colon \mathfrak{Rev}^{\mathcal{D}}(\mathfrak{M}) & \to & \mathfrak{Set} \\ Y & \mapsto & \operatorname{Hom}_{\operatorname{Frac}(\mathfrak{M})}(\operatorname{Frac} Y, \operatorname{Frac}(\widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x})) \end{array}$$

where Frac *denotes the ring of fraction.*

Note that unlike the classical Grothendieck-Murre theory, the base point is here supposed to belong to the normal crossing divisor.

The following result is essentially Theorem 3.7 of [23].

Theorem 3.4. *Let $\mathfrak{M}$ be a Deligne-Mumford k-stack, x be a smooth k-point of $\mathfrak{M}$ having a Nisnevich neighbourhood, $\mathcal{D}$ be a normal crossing divisor on $\mathfrak{M}$ and $t_{\mathcal{D}}$ a system of parameters of $\mathcal{D}$ at x. Then the* tangential base point *functor $F_x^{t_{\mathcal{D}}}$ is a* fibre functor*.*

The proof of this theorem goes by showing that this functor is isomorphic to a functor defined by a geometric point $x' \in \mathfrak{M}$, which is then a fibre functor by the theory of étale fundamental group. Here x' is given by the generic point of $\mathrm{Frac}(\widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x})$, which is an algebraic closure of $\mathrm{Frac}(\mathcal{O}^h_{\mathfrak{M},x})$ since $\mathrm{char}(k) = 0$.

In particular, this fiber functor gives an *arithmetic tame fundamental group based at a k-rational tangential base point* $\pi_1^{\mathcal{D}}(\mathfrak{M}; t_{\mathcal{D}}, x)$ defined as automorphism group of these tangential fiber functors. By base change, this arithmetic fundamental group admits a *geometric tame fundamental group* $\pi_1^{\mathcal{D}\times\bar{k}}(\mathfrak{M}\times\bar{k}; t_{\mathcal{D}\times\bar{k}}, x)$.

3.1.2. Consider the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ of k. Through universal properties, the ring $\widehat{\mathcal{O}}^h_{\mathfrak{M},x}\widehat{\otimes}\bar{k}$ inherits an action of $\mathrm{Gal}(\bar{k}/k)$, and so does $\widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x}$ since the system of parameters $t_{\mathcal{D}}$ is defined over k. From this follows that the functors $F_x^{t_{\mathcal{D}}}$ and $\pi_1^{\mathcal{D}}(\mathfrak{M}; t_{\mathcal{D}}, x)$ are both $\mathrm{Gal}(\bar{k}/k)$-equivariant:

Proposition 3.5. *Let $\mathfrak{M}$ be a k-algebraic stack, and $(x, t_{\mathcal{D}})$ be a k-rational tangential base point. Then there exists a* tangential *Gal$(\bar{k}/k)$-representation:*

$$\rho_{\vec{s}}\colon Gal(\bar{k}/k) \longrightarrow \mathrm{Aut}[\pi_1^{\mathcal{D}\times\bar{k}}(\mathfrak{M}\times\bar{k}; t_{\mathcal{D}\times\bar{k}}, x)].$$

Remark that a tangential base point is functorial through base change of k, whereas this is not the case for general morphisms between k-stacks: consider a morphism $f\colon \mathfrak{N} \to \mathfrak{M}$ of Deligne-Mumford k-stacks, $\mathcal{D}$ a normal crossing divisor on $\mathfrak{M}$, and $x \in \mathcal{D}(\mathrm{Spec}\, k)$ with $y \in f^{-1}(x)$ a k-rational point. Suppose that x and y both smooth and admit a Nisnevich neighbourhood.

Following lemma 2.9 of [23] we have:

$$\mathrm{Hom}_{\mathrm{Frac}(\mathfrak{M})}(\mathrm{Frac}\, Y, \mathrm{Frac}(\widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x})) = \mathrm{Hom}_{\mathfrak{M}}(Y, \widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x})$$

so we alternatively define a tangential base point with rings instead of fields. Now, suppose also that there are two systems of parameters $t_{f^*\mathcal{D}} = t'_1, \ldots, t'_n$ and $t_{\mathcal{D}} = \{t_1, \ldots, t_\ell\}$ of $f^*\mathcal{D}$ at y and $\mathcal{D}$ at x such that the induced morphism $f^h\colon \mathcal{O}^h_{\mathfrak{M},x} \to \mathcal{O}^h_{\mathfrak{N},y}$ sends t'_j to t_j or 0. The $\mathrm{Gal}(\bar{k}/k)$-equivariant natural transformation of functors

$$F_f\colon F_x^{t_{\mathcal{D}}} \to F_y^{t_{f^*\mathcal{D}}},$$

is then obtained through the $\mathrm{Gal}(\bar{k}/k)$-equivariant morphism

$$\widetilde{f}\colon \widetilde{\mathcal{O}}^{t_{\mathcal{D}}}_{\mathfrak{M},x} \to \widetilde{\mathcal{O}}^{t_{f^*\mathcal{D}}}_{\mathfrak{N},y}.$$

In the two important cases of unramified and smooth morphisms, an extension property guarantees the following nearly "functorial" result.

Proposition 3.6. *Let $f\colon \mathfrak{N} \to \mathfrak{M}$ be a morphism of Deligne-Mumford k-stacks, $\mathcal{D}$ a normal crossing divisor on $\mathfrak{M}$ such that $f^*\mathcal{D}$ is a normal crossing divisor on $\mathfrak{N}$, let $x \in \mathcal{D}(\mathrm{Spec}k)$, $y \in f^{-1}(x)$ a k-rational point and suppose that both x and y are smooth and have Nisnevich neighbourhood.*

If f is either smooth or unramified, then there exist regular system of parameters $t_{f^*\mathcal{D}}$ and $t_{\mathcal{D}}$ of $f^*\mathcal{D}$ at y and $\mathcal{D}$ at x and a Galois equivariant morphism

$$\pi_1^{f^*\mathcal{D}}(\mathfrak{N}; t_{f^*\mathcal{D}}, y) \to \pi_1^{\mathcal{D}}(\mathfrak{M}; t_{\mathcal{D}}, x).$$

Proof. Following the discussion above, it is sufficient to establish that there exists two system of parameters $t_{\mathcal{D}}$ for $\mathcal{D}$ at x and $t_{f^*\mathcal{D}}$ for $f^*\mathcal{D}$ at y such that f induces a morphism $f^h\colon \mathcal{O}^h_{\mathfrak{M},x} \to \mathcal{O}^h_{\mathfrak{N},y}$ sending an element of $t_{\mathcal{D}}$ on an element of $t_{f^*\mathcal{D}}$ or 0.

Consider the case f unramified, then the morphism $f^\#\colon \widehat{\mathcal{O}}^h_{\mathfrak{M},x} \to \widehat{\mathcal{O}}^h_{\mathfrak{N},y}$ is a surjection. Consider $t_1, \dots, t_n$ a system of parameters of $\mathcal{D}$ in $\widehat{\mathcal{O}}^h_{\mathfrak{M},x}$. As $f^*\mathcal{D} \neq \emptyset$, we have $f^h(t_1 \cdots t_\ell) \neq 0$ so that we can extract from $f^\#(t_1), \dots, f^\#(t_n)$ a system of generators of $\widehat{\mathcal{O}}_{\mathfrak{N},y}$ because $f^\#$ is formally unramified and induces an injection on tangent spaces.

Now consider the case f smooth. Then the morphism $f^\#$ is injective and we complete a system of parameters $t_1, \dots, t_n$ for $\mathcal{D}$ in $\widehat{\mathcal{O}}^h_{\mathfrak{M},x}$ into a system of parameters $t_1, \dots, t_{n'}$ of $\widehat{\mathcal{O}}^h_{\mathfrak{N},y}$ by picking up vectors in the tangent space. □

The lack of functoriality comes from the fact that *there is no obvious choice of parameters*, which has the important consequence as noted below.

Remark 3.7. A k-rational change of parameters $t_{\mathcal{D}}$ to $t'_{\mathcal{D}}$ – or *infinitesimal homotopic transformation* – leads to two k-*homotopically* equivalent k-rational base points $F_x^{t_{\mathcal{D}}} \simeq F_x^{t'_{\mathcal{D}}}$, but *not to equivalent* $\mathrm{Gal}(\bar{k}/k)$-*actions* on the fundamental groups: the action on Puiseux series makes some Kummer characters to appear from the N-th roots of the involved rational coefficients.

3.1.3. We now investigate conditions under which the choice of a tangential base point on $\mathfrak{M}$ defines a $\mathrm{Gal}(\bar{k}/k)$-action on the stack inertia groups of $\mathfrak{M}$. Let $\rho_{\vec{s}}\colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}[\pi_1^{\mathcal{D}\times\bar{k}}(\mathfrak{M} \times \bar{k}; \vec{s})]$ be a Galois representation defined by a k-tangential base point $\vec{s}$ on $\mathfrak{M}$ as in Proposition 3.5. Consider w a geometric point of $\mathcal{M}$ and let I_w denote its inertia group of 2-transformations - or *hidden paths*. Since a hidden path of I_w induces a transformation of the fiber functor F_w, this defines a morphism $\omega_w\colon I_w \to \pi_1(\mathfrak{M} \times \bar{k}, w)$ as in [19] §4.

Consider now $\bar{z}\colon \mathrm{Spec}\,(\bar{K}) \to \mathfrak{M}$ a geometric point, and choose an injection $\bar{k} \subset \bar{K}$. Since any $\sigma \in \mathrm{Gal}(\bar{k}/k)$ extends to a k-automorphism $\widetilde{\sigma}$ of $\bar{K}$, we define $\widetilde{\sigma}(\bar{z})$ by base change. Let us then choose two étale paths from $\bar{z}$ to $\vec{s}$ and from $\widetilde{\sigma}(\bar{z})$ to $\vec{s}$, thus defining morphisms $\phi\colon \pi_1(\mathfrak{M}, \bar{z}) \to \pi_1^{\mathcal{D}}(\mathfrak{M}; \vec{s})$ and $\phi_{\widetilde{\sigma}}\colon \pi_1(\mathfrak{M}, \widetilde{\sigma}(\bar{z})) \to \pi_1^{\mathcal{D}}(\mathfrak{M}; \vec{s})$.

The compatibility between σ and $\widetilde{\sigma}$ induces a diagram

$$
\begin{array}{ccccc}
I_{\bar{z}} & & \xrightarrow{\tau \mapsto \widetilde{\sigma}^{-1} \tau \widetilde{\sigma}} & & I_{\widetilde{\sigma}(\bar{z})} \\
\downarrow{\scriptstyle \omega_{\bar{z}}} & & & & \downarrow{\scriptstyle \omega_{\widetilde{\sigma}(\bar{z})}} \\
\pi_1(\mathfrak{M} \times \bar{k}, \bar{z}) & & & & \pi_1(\mathfrak{M} \times \bar{k}, \bar{z}) \\
& \searrow^{\phi} & & \nearrow^{\phi_{\widetilde{\sigma}}^{-1}} & \\
& \pi_1^{\mathcal{D} \times \bar{k}}(\mathfrak{M} \times \bar{k}; \vec{s}) & \xrightarrow{\sigma} & \pi_1^{\mathcal{D} \times \bar{k}}(\mathfrak{M} \times \bar{k}; \vec{s}) &
\end{array} \tag{3.1}
$$

which is commutative up to conjugacy by a hidden path from $\bar{z}$ to $\widetilde{\sigma}(\bar{z})$.

Note that when $\bar{z}$ is stable (as in Proposition 3.8) under $\mathrm{Gal}(\bar{k}/k)$, we can take $\phi_{\widetilde{\sigma}} = \phi$ for all σ in Diagram (3.1), which is then really commutative, i.e. without having to conjugate by some hidden path from $\bar{z}$ to $\bar{z}$. In particular, the tangential $\mathrm{Gal}(\bar{k}/k)$-action $\rho_{\vec{s}} \colon \sigma \mapsto \phi^{-1}\,\sigma\,\phi$ on $\pi_1^{\mathcal{D} \times \bar{k}}(\mathfrak{M} \times \bar{k}, \bar{z})$ sends the image $\omega_{\bar{z}}(I_{\bar{z}})$ into itself, and so induces an action $\rho^I_{\vec{s}}$ of $\mathrm{Gal}(\bar{k}/k)$ on $I_{\bar{z}}$.

Proposition 3.8. *Let $z \colon \mathrm{Spec}\, K \to \mathfrak{M}$ be a K-point, $\bar{z} \colon \mathrm{Spec}\, \bar{K} \to \mathfrak{M}$ a geometric point above z and $I_{\bar{z}} \to \pi_1(\mathfrak{M} \times \bar{k}, \bar{z})$ its stack inertia. Suppose that K/k is linearly disjoint from $\bar{k}/k$. Then any Galois representation $\rho_s \colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}[\pi_1^{\mathcal{D} \times \bar{k}}(\mathfrak{M} \times \bar{k}; \vec{s})]$ defined by a k-tangential base point $\vec{s}$ on $\mathfrak{M}$ defines a $\mathrm{Gal}(\bar{k}/k)$-action $\rho^I_{\vec{s}}$ on $I_{\bar{z}}$ and this action coincide with the action $\rho^I_{\bar{z}}$ of $\mathrm{Gal}(\bar{K}/K)$ on $I_{\bar{z}}$ by conjugacy.*

Proof. This follows from the discussion above, since K/k is linearly disjoint from $\bar{k}/k$, the $\bar{k}$-image of $\bar{z} \colon \mathrm{Spec}\, \bar{K} \to \mathfrak{M}$ is stable under the $\mathrm{Gal}(\bar{k}/k)$-action. The tangential $\mathrm{Gal}(\bar{k}/k)$-action $\rho_{\vec{s}} \colon \sigma \mapsto \phi^{-1}\,\sigma\,\phi$ on $\pi_1^{\mathcal{D} \times \bar{k}}(\mathfrak{M} \times \bar{k}, \bar{z})$ then sends the image $I_{\bar{z}}$ into itself, and so induces an action $\rho^I_{\vec{s}}$ of $\mathrm{Gal}(\bar{k}/k)$ on $I_{\bar{z}}$ according to the commutativity of the Diagram (3.1).

This proves furthermore, that $\rho^I_{\bar{z}}$ and $\rho^I_{\vec{s}}$, seen as an action of $\mathrm{Gal}(\bar{K}/K)$ through the surjection $\mathrm{Gal}(\bar{K}/K) \to \mathrm{Gal}(\bar{k}/k)$, define the same action on $I_{\bar{z}}$. □

Remark 3.9. Note that in case a morphism $f \colon \mathfrak{N} \to \mathfrak{M}$ and a point $y \in \mathfrak{N}(\mathrm{Spec}\, \bar{K})$ with image $x \in \mathfrak{M}(\mathrm{Spec}\, \bar{K})$ are given, together with all the compatible datas as in Proposition 3.6, the compatible $\mathrm{Gal}(\bar{k}/k)$-representations in $\pi_1^{f^*\mathcal{D}}(\mathfrak{N}; t_{f^*\mathcal{D}}, y)$ and $\pi_1^{\mathcal{D}}(\mathfrak{M}; t_{\mathcal{D}}, x)$ induce $\mathrm{Gal}(\bar{k}/k)$-actions on the inertia groups I_x and I_y by commutativity of Diagram (3.1).

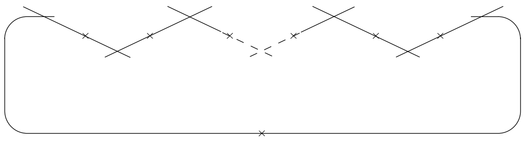
The situation of Proposition 3.8 and the compatibility through Knudsen morphism are applied in various situations to the moduli spaces of curves in §4.

3.2 Tangential Galois Action and Clutching Morphisms

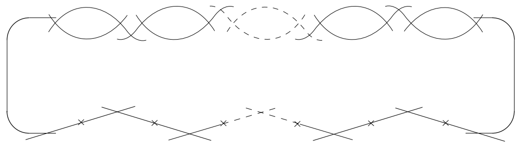
We now clarify the tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action results of the previous section in the case of the Deligne-Mumford $\mathbb{Q}$-stack of the moduli spaces of stable curves $\bar{\mathcal{M}}_{g,[m]}$ and of Knudsen morphisms. The case of the stack inertia requiring an additional specific results is treated in §4.1.1 and §4.1.2.

3.2.1. We first detail the choice of tangential base point in $\mathcal{M}_{g,[m]}$. Let $x \in \bar{\mathcal{M}}_{g,[m]}(\mathrm{Spec}\,\mathbb{Q})$ be a maximally degenerated $\mathbb{Q}$-curve defined as a graph of $\mathbb{P}^1$ such that marked and singular points are rational, so that x has only rational automorphisms. By Proposition 3.2, $x \in \bar{\mathcal{M}}_{g,[m]}(\mathbb{Q})$ admits a Nisnevich neighbourhood.

Note that as an example of such a curve, one can take those of [8, Fig. $(ii)_n$, $(iii)'_{k,n}$] reproduced in Fig. 2 below for $g \geqslant 1$, and which we denote by X_A and X_B.



(A) Curve X_A - type $g = 1, n \geqslant 1$



(B) Curve X_B - type $g \geqslant 2, n \geqslant 1$

FIGURE 2. Maximally degenerated curves of type (g, n)

Remark 3.10. The original approach of [8] by *tangential base point on* $\mathcal{M}_{g,[m]}$ is complementary to our construction in §3.1.1: they define a maximally degenerated curve X by a $\mathbb{P}^1_{0,1,\infty}$-diagram and then fix a canonical choice of a set of coordinates $\mathbf{q}$ of the universal deformation ring $\mathcal{O}_X^{def}$ of X. This corresponds exactly to the choice of a system of parameters which led to the definition 3.3.

3.2.2. Let us consider the Knudsen's clutching morphisms between moduli spaces of stable curves as defined in [9]:

$$\beta\colon \bar{\mathcal{M}}_{g-1,m+2} \longrightarrow \bar{\mathcal{M}}_{g,m}.$$

We now define the *partial compactification of* $\mathcal{M}_{g,[m]}$ *relatively to* β. Let $\mathcal{E} = \bar{\mathcal{M}}_{g,[m]} \setminus \mathcal{M}_{g,[m]}$, and $\mathcal{D}$ be the closure of $\mathcal{E} \setminus \mathrm{Im}(\beta)$ in $\bar{\mathcal{M}}_{g,[m]}$, which is the union of the irreducible components of $\mathcal{E}$ not containing $\mathrm{Im}(\beta)$. Let $\bar{\mathcal{M}}_{g-1,[m]+2}$ denote the moduli spaces of stable curves with m marked points and 2 fixed points. Then $\mathcal{D}$ is a normal crossing divisor in $\bar{\mathcal{M}}_{g,[m]}$ and $\beta^*(\mathcal{D})$ is a normal crossing divisor in $\bar{\mathcal{M}}_{g-1,[m]+2}$ equal to $\bar{\mathcal{M}}_{g-1,[m]+2} \setminus \mathcal{M}_{g-1,[m]+2}$. The partial compactification is then defined as $\widetilde{\mathcal{M}}_{g,[m]} = \bar{\mathcal{M}}_{g,[m]} \setminus \mathcal{D}$, and we still denote by β the induced Knudsen morphism from $\bar{\mathcal{M}}_{g-1,[m]+2}$ to $\widetilde{\mathcal{M}}_{g,[m]}$.

Note that the partial normalisation X'_A (resp. X'_B) of X_A (resp. X_B) at a singular point P, and pointed at the pre-images of P, is then naturally a curve in $\bar{\mathcal{M}}_{g-1,[m]+2}$ sent to X_A (resp. X_B) by β.

In the following, the fundamental group is based at a point given by X_A or X_B in the *partial compactification of* $\widetilde{\mathcal{M}}_{g,[m]}$. A particular choice of a tangential base point $\vec{s} = (x, t_{\mathcal{D}})$ based at X_A or X_B, leads to the fundamental group

$\pi_1^{\mathcal{D}}(\bar{\mathcal{M}}_{g,[m]};\vec{s})$ that we denote by $\pi_1(\widetilde{\mathcal{M}}_{g,[m]};\vec{s})$. Thanks to Theorem 3.4, the fundamental group $\pi_1(\widetilde{\mathcal{M}}_{g,[m]};\vec{s})$ is isomorphic to $\pi_1(\widetilde{\mathcal{M}}_{g,[m]};x)$ for any geometric point $x \in \widetilde{\mathcal{M}}_{g,[m]}$. In the same way, for a β-compatible tangential base point $\vec{s}\,'$ we denote by $\pi_1(\mathcal{M}_{g-1,[m]+2};\vec{s}\,')$ the fundamental group $\pi_1^{\beta*(\mathcal{D})}(\bar{\mathcal{M}}_{g-1,[m]+2};\vec{s}\,')$.

3.2.3. Recall that the $\mathbb{Q}$-stack $\mathcal{M}_{g,[m]}$ having the Artin-Mazur étale homotopy type of a $K(\pi,1)^\wedge$ space by [20], the étale fundamental group associated to a geometric point x of $\mathcal{M}_{g,[m]}$ yields to an Arithmetic-Geometric (short) Exact Sequence:

$$1 \to \pi_1(\mathcal{M}_{g,[m]} \times \bar{\mathbb{Q}}, x) \to \pi_1(\mathcal{M}_{g,[m]}, x) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1.$$

For a given normal crossing divisor $\mathcal{D}$ and a given $\mathbb{Q}$-tangential base point $\vec{s} = (x, t_{\mathcal{D}})$ on $\bar{\mathcal{M}}_{g,[m]}$, Proposition 3.5 yields a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation:

$$\rho_{\vec{s}}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}\left(\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \times \bar{\mathbb{Q}};\vec{s})\right). \tag{3.2}$$

where $\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \times \bar{\mathbb{Q}};\vec{s})$ denotes $\pi_1^{\mathcal{D}\times\bar{\mathbb{Q}}}(\bar{\mathcal{M}}_{g,[m]} \times \bar{\mathbb{Q}};\vec{s})$.

With the notation of 3.2.2, we can construct β-compatible representations $\rho_{\vec{s}}$ using Proposition 3.6 since β is *unramified* by [9] Corollary 3.9.

Proposition 3.11. *There exist $\mathbb{Q}$-tangential base points $\vec{s}$ and $\vec{s}'$ respectively on $\bar{\mathcal{M}}_{g,m}$ and $\bar{\mathcal{M}}_{g-1,m+2}$, with a morphism:*

$$\pi_1(\mathcal{M}_{g-1,m+2};\vec{s}') \longrightarrow \pi_1(\widetilde{\mathcal{M}}_{g,m};\vec{s})$$

which is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-equivariant, the support of $\vec{s}$ and $\vec{s}\,'$ being of type X_A or X_B.*

We insist on the fact that the Knudsen morphisms do not leads to *canonical* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions – see Remark 3.7. The comparison of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action by change of parameters illustrates the non $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariance of *analytic continuation*. It is *the core of the Arithmetic Geometry of moduli spaces of curves* as illustrated by the role of Deligne's *droit chemin p from* $\vec{01}$ *to* $\vec{10}$ in $\mathcal{M}_{0,4}$ as in [7].

In §4.2.3, we need the following specific case of the Proposition above.

Corollary 3.12. *Let $\beta_\Diamond\colon \bar{\mathcal{M}}_{g-1,[m]+2} \to \bar{\mathcal{M}}_{g,[m]}$ be the clutching morphism induced by β. Then there exist $\vec{s}$ and $\vec{s}'$ compatible tangential base points respectively on $\bar{\mathcal{M}}_{g,[m]}$ and $\bar{\mathcal{M}}_{g-1,[m]+2}$ with a morphism:*

$$\pi_1(\bar{\mathcal{M}}_{g-1,[m]+2};\vec{s}') \longrightarrow \pi_1(\bar{\mathcal{M}}_{g,[m]};\vec{s})$$

which is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-equivariant.*

Proof. Consider the cartesian diagram

$$\begin{array}{ccc} \bar{\mathcal{M}}_{g-1,m+2} & \xrightarrow{\ \beta\ } & \bar{\mathcal{M}}_{g,m} \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ \bar{\mathcal{M}}_{g-1,[m]+2} & \xrightarrow{\ \beta_\Diamond\ } & \bar{\mathcal{M}}_{g,[m]} \end{array}$$

where vertical morphisms are étale since the marked points are supposed distinct. Then $\beta_\Diamond$ is unramified since the unramified property is local at the source for the étale topology by descent property. The result then follows from Proposition 3.6. □

Remark 3.13. As a special case and as another general application of Proposition 3.6, we signal the following:

(1) This construction of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations compatible with β is an algebraic generalisation of the approach of [2] where a topological morphism $\Gamma^2_{0,[m]} \to \Gamma_{1,[m]}$ is defined between *mapping class groups* to deal with the étale type inertia in genus 1;

(2) The Knudsen's morphisms

$$\beta_{g_1,g_2}\colon \bar{\mathcal{M}}_{g_1,m_1} \times \bar{\mathcal{M}}_{g_2,m_2} \longrightarrow \bar{\mathcal{M}}_{g,m}.$$

being closed immersions, the approach above readily applies to the study of various β_{g_1,g_2}-compatible $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations.

4 Galois Action on Inertia

We establish the Galois action result on the cyclic stack inertia of $\mathcal{M}_{g,[m]}$. We follow the geometric approach through irreducible components of *special loci* as initiated in [4], which we first recall and complete by precising the definition of the Galois action on stack inertia groups. We then discuss the behaviour of this Galois action under specialisation, and establish the main theorem which results from the previous sections.

4.1 Special Loci and Inertia Groups

For G a stack inertia group of $\mathcal{M}_{g,[m]}$, we consider the *special loci* $\mathcal{M}_{g,[m]}(G)$ associated to G which is defined as the locus of points of $\mathcal{M}_{g,[m]}$ whose automorphisms group contains a subgroup isomorphic to G. Consider $\rho_s\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}[\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \times \bar{\mathbb{Q}}; \vec{s})]$ a Galois representation defined by a $\mathbb{Q}$-tangential base point $\vec{s}$ as in §3.2.3. Let w be a geometric point of $\mathcal{M}_{g,[m]}$ and denotes by I_w its inertia group of *hidden paths*. By the residual finitness property of the orbifold fundamental group for moduli spaces of curves, one obtains an *injective* morphism $\omega_w\colon I_w \hookrightarrow \pi_1(\mathcal{M}_{g,[m]} \times \bar{\mathbb{Q}}, w)$ whose $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-properties are discussed below.

4.1.1. Let G be a finite *cyclic* subgroup of $\pi_1^{orb}(\mathcal{M}_{g,[m]}(\mathbb{C}))$, and let $\underline{\mathbf{kr}}$ be the *algebraic branching data* defined in [4] and which are proven to characterise the *irreducible* Deligne-Mumford substacks $\mathcal{M}_{g,[m],\underline{\mathbf{kr}}}(G)$ of $\mathcal{M}_{g,[m]}(G)$. By the study of the normalisation $\mathcal{M}_{g,[m]}[G]/\mathrm{Aut}(G)$ of $\mathcal{M}_{g,[m]}(G)$, we prove every irreducible component to be defined over $\mathbb{Q}$ and to be geometrically irreducible – see Theorem 4.3 Ibid. From this fundamental result, we deduce the following lemma.

Lemma 4.1 ([4] Lemma 5.2)**.** *For any irreducible component* $Z \subset \mathcal{M}_{g,[m]}(G)$ *there exists a morphism* $\mathrm{Spec}\, K \to Z$ *with* K *linearly disjoint from* $\mathbb{Q}$*.*

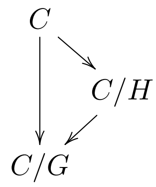
Let I be the generic inertia group of an irreducible component Z of the special loci $\mathcal{M}_{g,[m]}(G)$. Lemma 4.1 gives a K-point z of the component whose geometric inertia $I_{\bar{z}}$ contains I. Proposition 3.8 allows us to read the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action given by $\rho_{\vec{s}}$ on $I < I_z$ through the $\mathrm{Gal}(\bar{K}/K)$-action given by conjugacy.

Note that the definition of this action relies on many choices as to fix an algebraic closure of K, and comes in fact from the identification of two Galois actions $\rho^I_{\vec{s}}$ and $\rho^I_{\bar{z}}$, see Proposition 3.8 for details.

Remark 4.2. In Lemma 4.1, the K-point of an irreducible component is built by factorisation through a certain base change of $\mathcal{M}_{g,[m]}$ in order to kill the automorphisms of the gerbe at the generic point. In particular, when the irreducible component admits a dense open subset with trivial automorphism group, $\operatorname{Spec} K \to Z$ is the generic point of the component.

4.1.2. For a general curve $C \in \mathcal{M}_{g,[m]}(G)$, recall that the associated G-cover $C \to C/G$ factorises as below with the properties:

(1) the group $H < G$ is generated by the stabilisers of ramification points of the G-cover $C \to C/G$;
(2) the cover $C/H \to C/G$ is étale.



When $H = G$, the action of G on C is said to be *without étale factorisation*. In this case, the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on stack inertia is specified by the Proposition below, which also plays a key role in the final proof of the general case.

Proposition 4.3. *Let $\eta\colon \operatorname{Spec} K \to \mathcal{M}_{g,[m]}(G)$ be a morphism with value into a field K linearly disjoint from $\mathbb{Q}$. Suppose that the curve η is without étale factorisation. Then for $\sigma \in \mathrm{Gal}(\bar{K}/K)$ and $\gamma \in G$ we have $\sigma.\gamma = \gamma^{\chi(\sigma)}$.*

Let $C\colon \operatorname{Spec} K \to \mathcal{M}_{g,[m],\underline{\mathbf{kr}}}(G)$ be a morphism as in Lemma 4.1, and suppose that the action of G on the curve C is without étale factorisation. Since the stabilisers of a ramification point are generating subgroups of the stack inertia group, the *branch cycle argument* implies that the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is given by $\sigma.\gamma = \gamma^{\chi(\sigma)}$ on a generator γ of the inertia group. We refer to Theorem 5.4 of [4] for details.

In the following section, the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-compatibility of the Knudsen morphism of §3.2.3 allows us to deduce the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the inertia group of a *curve with possible étale factorisation* from the case of a curve with no étale factorisation as treated above.

4.2 Inertial Limit Galois Action and Cyclotomy

We extend the cyclotomy result from curves without étale factorisation to general curves. This relies on the behaviour of stack inertia along degeneracy as in §2.2.3 and on the compatible $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations of §3.2.3, which reduces the study of the stack inertia from $\mathcal{M}_{g,[m]}(G)$ to a specific locus of curves in $\mathcal{M}_{g-1,[m]+2}(G)$.

This process leads to the notion of an *inertial limit Galois action*. Note that the results of the two first paragraphs readily extends to any Deligne-Mumford stack.

4.2.1. We first consider the behaviour of stack inertia groups under the process of specialisation. The following result should be read from the points of view of the result of generic G-deformation of a smooth curve of §2.2 and in relation with the Galois actions defined by a tangential base point in the partial compactification $\widetilde{\mathcal{M}}_{g,[m]}$ of §3.2.3.

Lemma 4.4. *Let η and z be two points in $\widetilde{\mathcal{M}}_{g,[m]}$ such that z is a specialisation of η. Let R be a valuation ring and $T \in \widetilde{\mathcal{M}}_{g,[m]}(\operatorname{Spec} R)$ whose generic fibre is a geometric point $\bar{\eta}$ above η and whose special fibre is a geometric point $\bar{z}$ above z. Then the choice of T induces an étale path $\bar{\eta} \rightsquigarrow \bar{z}$ which sends the stack inertia $I_{\bar{\eta}}$ into $I_{\bar{z}}$.*

Proof. Since étale coverings are proper morphisms, the choice of T defines an étale path $\phi_{\bar{\eta}\rightsquigarrow\bar{z}}$ from $\bar{\eta}$ to $\bar{z}$ by using the valuative criterion for properness. This choice is by definition compatible with specialisation.

Consider the curves $C_{\bar{\eta}}$ and $C_{\bar{z}}$ with their respective automorphism groups $\operatorname{Aut}(C_{\bar{\eta}})$ and $\operatorname{Aut}(C_{\bar{z}})$. The stable reduction process induces a morphism $\phi\colon \operatorname{Aut}(C_{\bar{\eta}}) \to \operatorname{Aut}(C_{\bar{z}})$, where ϕ is injective thanks to the non-ramification of the diagonal of $\bar{\mathcal{M}}_{g,[m]}$. The lemma follows from the commutativity of the diagram

$$\begin{array}{ccc} \operatorname{Aut}(C_{\bar{\eta}}) = I_{\bar{\eta}} & \hookrightarrow & \pi_1(\widetilde{\mathcal{M}}_{g,[m]}, \bar{\eta}) \\ \downarrow \phi & & \downarrow \bar{\eta}\rightsquigarrow\bar{s} \\ \operatorname{Aut}(C_{\bar{z}}) = I_{\bar{z}} & \hookrightarrow & \pi_1(\widetilde{\mathcal{M}}_{g,[m]}, \bar{z}). \end{array}$$

□

4.2.2. Considering a $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant point η of an irreducible component $\mathcal{M}_{g,[m],\underline{\mathbf{kr}}}(G)$ – as for example given by Lemma 4.1 – we now discuss how the two compatible $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions on the fundamental group and the stack inertia group I_η of §4.1.1 behave according to a specialisation of η.

Let us consider a $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation $\rho_{\vec{s}}$ on $\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \times \bar{\mathbb{Q}}, \vec{s})$ as given by a tangential base point $\vec{s}$ on $\widetilde{\mathcal{M}}_{g,[m]}$, and let $z \in \widetilde{\mathcal{M}}_{g,[m],\underline{\mathbf{kr}}}(G)$ be a specialisation of η. More precisely, let R be a valuation ring with algebraically closed fraction field and residue field, endowed with a morphism $T\colon \operatorname{Spec} R \to \mathcal{M}_{g,[m]}(G)$ which sends the generic point of $\operatorname{Spec} R$ onto η and the closed point on z – thus defining two geometric points $\bar{\eta}$ and $\bar{z}$. Let also $\phi_{\bar{\eta}}$ (resp. $\phi_{\bar{z}}$) be an étale path from $\bar{\eta}$ to $\vec{s}$ (resp. from $\bar{z}$ to $\vec{s}$) as given by change of base point in $\widetilde{\mathcal{M}}_{g,[m]}$. Since étale coverings are proper morphisms, the choice of T defines an étale path $\phi_{\bar{\eta}\rightsquigarrow\bar{z}}$ from $\bar{\eta}$ to $\bar{z}$, and following Diagram 4.1 below:

$$
\begin{array}{ccccc}
I_{\bar z} & & & & I_{\bar\eta} \\
\downarrow{\scriptstyle \omega_{\bar z}} & & & & \downarrow{\scriptstyle \omega_{\bar\eta}} \\
\pi_1(\widetilde{\mathcal{M}}_{g,[m]}, \bar z) & & \xrightarrow{\ \phi_{\bar\eta\rightsquigarrow\bar z}\ } & & \pi_1(\widetilde{\mathcal{M}}_{g,[m]}, \bar\eta) \\
& \searrow{\scriptstyle \phi_{\bar z}} & & \swarrow{\scriptstyle \phi_{\bar\eta}} & \\
& & \pi_1(\widetilde{\mathcal{M}}_{g,[m]}; \vec{s}) & &
\end{array}
\tag{4.1}
$$

we obtain an isomorphism $\phi_{\bar z}(I_{\bar z}) \simeq \phi_{\bar\eta}(I_{\bar\eta})$ as subgroups of $\pi_1(\widetilde{\mathcal{M}}_{g,[m]}; \vec{s})$.

When the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the stack inertia group of the generic point $I_{\bar\eta}$ is given by cyclotomy– as for example in Corollary 4.6 in the case of non-étale factorisation – this allows the computation of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the stack inertia group of the specialisation $I_{\bar z}$.

Lemma 4.5. *Let η be a generic point of an irreducible component of a special loci $\mathcal{M}_{g,[m]}(G)$, and denote by z a specialisation of η. If there exists a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on $G \subset I_{\bar\eta}$ given by cyclotomy, then the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on $G \subset I_{\bar z}$ is given by χ-conjugacy.*

Proof. Denote by τ a generator of $I_{\bar z}$ (resp. γ generator of $I_{\bar\eta}$), and let $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, whose image by the given $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation $\rho_{\vec{s}}$ is still denoted by σ. Writing

$$\tau = \phi_{\bar\eta\rightsquigarrow\bar z} \circ \gamma \circ \phi^{-1}_{\bar\eta\rightsquigarrow\bar z},$$

leads to

$$\sigma.\tau = \rho_\sigma.\gamma^{\chi(\sigma)}.\rho_\sigma^{-1}$$

where $\rho_\sigma = \sigma(\phi_{\bar\eta\rightsquigarrow\bar z}) \circ \phi^{-1}_{\bar\eta\rightsquigarrow\bar z}$ is an étale path in $\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \times \bar{\mathbb{Q}}; \vec{s})$. □

In particular, for curves without étale factorisation, this implies the following extension of Proposition 4.3 using Proposition 3.8, which allows us to read the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action at the level of automorphism groups.

Corollary 4.6. *Let $\rho_{\vec{s}}$ be a tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation, and let G be a cyclic stack inertia group of $\mathcal{M}_{g,[m]}$ satisfying the non-étale factorization property. Then for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on an element $\gamma \in G$ is given by*

$$\sigma.\gamma = \rho_\sigma\, \gamma^{\chi(\sigma)}\, \rho_\sigma^{-1}$$

for ρ_σ an étale path of $\pi_1(\bar{\mathcal{M}}_{g,[m]} \times \bar{\mathbb{Q}}, \vec{s})$.

In the following, we say that such a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action on a stack inertia element *is given by χ-conjugacy.*

Remark 4.7. A few remarks on the importance of the factor of conjugacy of the Corollary above.

(1) The *droit chemin* p of $\mathcal{M}_{0,4}$ from $\vec{01}$ to $\vec{10}$ – see [7] – admits a factorisation by the path r from $\vec{01}$ to $1/2$, and under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action p gives rise to factors f_σ and r to g_σ – see [11]. Noting that $1/2$ represents a point of $\mathcal{M}_{0,[4]}$ with cyclic inertia $\mathbb{Z}/2\mathbb{Z}$, the cocycle $\rho_\sigma = \sigma(\phi_{\bar{\eta}\rightsquigarrow\bar{z}}) \circ \phi^{-1}_{\bar{\eta}\rightsquigarrow\bar{z}}$ plays a similar role to this factor g_σ.

(2) The case of a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation in the *outer automorphism group* is more troublesome: the conjugacy of a stack inertia group I by any étale path of $\pi_1(\mathcal{M}_{g,[m]} \times \bar{\mathbb{Q}})$ can partly cancel I, hence leading to a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on a proper and possibly trivial subgroup of I *only*.

4.2.3. We now establish the main result of the paper, which follows from all the results collected in the previous sections.

Theorem 4.8. *Let I be a cyclic stack inertia group of $\mathcal{M}_{g,[m]}$. Then there exists a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-representation in* $\pi_1(\mathcal{M}_{g,[m]} \times \bar{\mathbb{Q}})$ *such that* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts on I by χ-conjugacy.*

Proof. Following Corollary 4.6 the remaining case is when I is the automorphism group of a curve $C \in \mathcal{M}_{g,[m]}$ with étale factorisation; in the case $g = 0$ all the automorphisms of curves being without étale factorisation, we therefore suppose $g \geqslant 1$.

Let η be a generic point of the special loci $\mathcal{M}_{g,[m]}(I)$, and let z be a specialisation of η given by Corollary 2.7 as a stable I-curve. Let $\eta' \in \mathcal{M}_{g-1,[m]+2}(I)$ be the generic point of the component containing $\beta_\diamondsuit^{-1}(z)$. Then $\xi = \beta_\diamondsuit(\eta')$ is a specialisation of η and has a normalisation of genus $g-1$ without étale ramification as in Corollary 2.7.

Let us now fix two tangential $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations in $\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \times \bar{\mathbb{Q}}; \vec{s})$ and in $\pi_1(\mathcal{M}_{g-1,[m]+2} \times \bar{\mathbb{Q}}; \vec{s}\,')$ as given in Corollary 3.12, which are thus compatible to the Knudsen's morphism

$$\pi_1(\mathcal{M}_{g-1,[m]+2} \times \bar{\mathbb{Q}}, \vec{s}\,') \xrightarrow{\beta_\diamondsuit} \pi_1(\widetilde{\mathcal{M}}_{g,[m]} \times \bar{\mathbb{Q}}, \vec{s}),$$

and also at the level of stack inertia groups according to Remark 3.9. This hence reduces the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the stack inertia of $\mathcal{M}_{g,[m]}$ to the stack inertia of $\mathcal{M}_{g-1,[m]+2}$ as follows: η' being a generic curve without étale factorisation in $\mathcal{M}_{g-1,[m]+2}$, the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the inertia group $I_{\eta'}$ is given by χ-conjugacy by Corollary 4.6, then by χ-conjugacy on I_ξ in $\mathcal{M}_{g,[m]}$ by the compatibility of the action through $\beta_\diamondsuit$. By the property of injectivity under specialisation of Lemma 4.4, this then implies the same for the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on I viewed in the generic automorphism group $I_\eta < I_\xi$.

Since the canonical morphism $\pi_1(\mathcal{M}_{g,[m]} \times \bar{\mathbb{Q}}) \to \pi_1(\widetilde{\mathcal{M}}_{g,[m]} \times \bar{\mathbb{Q}})$ induces an injection at the level of stack inertias, the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action by χ-cyclotomy on I viewed as inertia group in $\pi_1(\widetilde{\mathcal{M}}_{g,[m]} \times \bar{\mathbb{Q}})$ then finally implies the same for I as inertia group in $\pi_1(\mathcal{M}_{g,[m]} \times \bar{\mathbb{Q}})$. □

4.2.4. Starting with the curve C of a given strata of the special loci $\mathcal{M}_{g,[m]}(G)$, we define two $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions on $\pi_1(\mathcal{M}_{g-1,[m]+2};\vec{s}\,')$ and $\pi_1(\widetilde{\mathcal{M}}_{g,[m]};\vec{s})$ which, as given in §4.1.1 and Corollary 3.12, are compatible at the level of stack inertia groups. By the above process of generisation and normalisation, it results in a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the automorphism group of a curve *which does not belong to the same inertia strata*. This resulting $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action hence deserves the name of *Inertial Limit Galois Action*.

Note that this comparison of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions on different type of stack inertia through a *clutching Knudsen morphism* can be seen as a *stack inertia* analogous to the *infinity inertia* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action comparison between degenerated strata which is based on the other *clutching morphism* $\bar{\mathcal{M}}_{g_1,m_1} \times \bar{\mathcal{M}}_{g_2,m_2} \to \bar{\mathcal{M}}_{g,m}$ – see [13].

The extension of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action by cyclotomy from the first inertia stratas ${}_1\mathcal{M}^{in}_{g,[m]}(\mathbb{Z}/p\mathbb{Z})$ for p prime, to the stratas ${}_k\mathcal{M}^{in}_{g,[m]}(\mathbb{Z}/n\mathbb{Z})$ associated to any cyclic group, motivates a more systematic study *from the point of view of the combinatoric of the inertia stratification* $\{{}_k\mathcal{M}^{in}_{g,[m]}(G)\}_k$. This may results in a complete description of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the whole second stratas ${}_2\mathcal{M}^{in}_{g,[m]}(G)$ for generic G, but also following a long Grothendieck-Teichmüller tradition to *new stack inertia* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-*equations* satisfied by the conjugacy factors ρ_σ of the action by χ-conjugacy .

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