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Homotopic and Geometric Galois Theory (online meeting)

Organized by
Benjamin Collas, Kyoto; Pierre Dèbes, Villeneuve d'Ascq; Hiroaki Nakamura, Osaka; Jakob Stix, Frankfurt

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Abstracts

Simplicial and Homotopical Aspects of Arithmetic Geometry

Benjamin Collas

Following the "Longue marche à travers la théorie de Galois" (1981) and the study of the absolute Galois group of rational via combinatorial group-theoretic properties of the moduli spaces of curves $\mathcal{M}_{g,[m]}$ – now Grothendieck-Teichmüller and anabelian geometry theories, Grothendieck's "À la poursuite des champs" (1983) lays the theoretical and *categorical* foundation for closing further the gap between algebraic topology and algebraic geometry – see now Lurie's ∞-categories and Toën's geometry. As early exploited by T. Oda, this includes the higher consideration of homotopy groups and stack symmetries of the arithmetic of the spaces.

This report reviews results and techniques from *Homotopical Arithmetic Geometry* in relation with the rational, motivic, and arithmetic aspects of the workshop (Topics 1, 2 and 3) such as led **(1)** by these higher consideration, and **(2)** by bringing closer spaces, anabelian and abelian geometries – see Fig. 1.

$$\begin{tikzcd}
\mathrm{Sch}_k \arrow[rr, bend left, "(-)_{\mathbb{A}^1}"] \arrow[dr, "\{-\}_{et}"] \arrow[dd] & & & \mathcal{SH}^{et}(\mathbb{Q}) \arrow[dd, shift left, "N"] \\
& \text{Pro-Sp} \arrow[r, shift right, "S"'] & \mathcal{H}^{et}(\mathbb{Q}) \arrow[l, shift right, "R_{et}"'] \arrow[ur, "\Sigma^\infty"] \arrow[dr, dashed] & \\
\mathrm{St}(k) \arrow[ur, dashed, "\{-\}_{et}"'] \arrow[urr, dashed, bend right, "(-)_{\mathbb{A}^1}"'] & & & DA^{et}(\mathbb{Q}) \arrow[uu, shift left, "\Gamma"]
\end{tikzcd}$$

Spaces | Anab. Geo. | $\mathbb{A}^1$-Geo. | Motives

Fig 1. Geometries, arithmetic and motives: Friedlander's pro-spaces, Morel-Voevodsky's (un)stable motivic homotopy, and Ayoub's motives.

1. Geometric Galois Action and Anabelian Geometry

Anabelian investigations rely on seminal properties of the arithmetic-geometric exact sequence **(AGS)** and the associated geometric Galois action **(GGA)**:

$$1 \leftarrow \mathrm{Spec}k \underset{\dashrightarrow}{\leftarrow} \pi_1^{et}(X,*) \leftarrow \pi_1^{et}(X,*) \leftarrow \pi_1^{et}(X_{\bar{k}},*) \ldots \quad \rightsquigarrow G_k \to \mathrm{Out}[\pi_1^{et}(X_{\bar{k}},*)].$$

Following Grothendieck's original insight, they are expressed in terms of group-theoretic properties (e.g. center-freeness, terminality of inertia-decomposition groups) for *topological* $K(\pi,1)$ varieties over number fields (Nakamura, Tamagawa) and p-adic fields (Mochizuki, Y. Hoshi) – see [IhNa97] for an overview.

1.1. Anabelian Geometry & Étale Homotopy Type.

For k sub-p-adic field, X a smooth connected variety and Y a hyperbolic curve over k, a reformulation of Mochizuki's Th. A in terms of $\{X\}_{et}$ is given by [SS16]:

$$\text{(1)} \qquad \{-\}_{et} \colon \mathrm{Isom}_K(X,Y) \underset{K(\pi,1)}{\xrightarrow{\sim}} \mathrm{Isom}_{\mathrm{Ho(Pro\text{-}Sp)}\downarrow ket}[\{X\}_{et},\{Y\}_{et}]$$

that follows the expected π_1-Hom-property between classifying pro-space BG and pro-group G, and relies on the centre-freeness of G_k.

In higher dimension, assuming a certain factor-dominant immersion $Y \hookrightarrow C_1 \times \cdots \times C_n$ into hyperbolic curves (HC) provides over number fields **(1)** the *existence of a functorial retract* of $\{-\}_{et}$ by application of Tamagawa's Lemma for separating rational points and of Lefschetz counting-points formula, that implies **(2)** that:

> *Every smooth variety over a number field K admits a relative anabelian Zariski basis $\mathcal{U}$,*

i.e. such that Eq. (1) is satisfied for any X, $Y \in \mathcal{U}$ – see ibid.

Note that the later relies on the existence of strongly hyperbolic Artin neighbourhoods in smooth varieties, i.e. for the scheme to be the abutment of some *elementary fibrations* into hyperbolic curves $\{X_i\}_{i=0,\ldots,n}$ that satisfy the (HC) property.

1.2. Higher Anabelian Topological Types.

For $\mathcal{M}$ Deligne-Mumford stacks, and for the moduli of curves $\mathcal{M}_{g,[m]}$ in particular, the stack inertias $I_{\mathcal{M},*} \to \mathcal{M}$ of *cyclic type* share similar properties with the divisorial (anabelian) ones: they are topological generators with a GGA Tate-type [CoMa14], and Serre's goodness is at once an Artin neighbourhood and a $I_{\mathcal{M}}$-property – that measures the discrepancy between topological and étale $K(\pi,1)$. This motivates *the investigation of stack anabelian obstructions in terms of* $\{\mathcal{M}\}_{et}$ – see $\{-\}_{et}$ in Fig. 1 and §2.

In terms of higher dimensional anabelian geometry, one could investigate some étale topological type Postnikov anabelian obstruction for well-chosen varieties over number fields. We also refer to techniques of p-adic fields which provides natural classes of anabelian varieties (e.g. of Belyi type) for absolute (and relative) versions of (2) above – see Y. Hoshi's report in this volume.

1.3. **An Anabelian $\mathbb{A}^1$-geometry?** It follows Isaksen's Quillen adjunction (R_{et}, S) – see also A. Schmidt's work – that $\{-\}_{et}$ factorizes through $(-)_{\mathbb{A}^1}$, thus motivating the investigation of *anabelian obstructions in terms of non-$\mathbb{A}^1$-rigid invariants.*

2. Motives for the Moduli Stack of Curves

Morel-Voevodsky's (unstable) motivic homotopy categories $\mathcal{H}^{et}(k)$ and $\mathcal{SH}^{et}(k)$ of simplicial presheaves – and spectras of – provides a functorial bridge between the arithmetic and motivic properties of spaces – see Fig. 1; from the category of stacks $\mathrm{St}(k)$ to Ayoub's derived weak Tannaka category $DA^{et}(k)$.

2.1. **Algebraic & Topological Circles for Stacks.** Jardine's Quillen categorical $\mathrm{St}(k) = \mathrm{Ho}[sPr_{et}(\mathrm{Aff}_k)]$ leads to the definition of motive for stack $M(\mathcal{M}) := N[\Sigma^\infty(M_{\mathbb{A}^1})]$ with the consequences of: **(1)** recovering the stack inertia as the derived $\mathbb{S}^1$-loop space $I_{\mathcal{M}} = [\mathbb{S}^1, \mathcal{M}]$ in the algebraic-topologic decomposition of the motivic Lefschetz $\mathbb{P}^1 = \mathbb{S}^1 \wedge \mathbb{G}_m$, and **(2)** considering a new homological fiber functor to bypass the pro-unipotent nature of the "scheme" mixed Tate motives.

2.2. **A Stack Inertia Decomposition.** Indeed, the Hochschild homological functor $\mathrm{HH}_\bullet$ applied to the GGA *arithmetic* stack inertia decomposition of the cyclic special loci $\mathcal{M}_{g,[m]}(\gamma)$ of [CoMa14] provides:

A motivic stack inertia decomposition for $\mathcal{M}_{g,[m]}$:

$$M(\mathcal{M}) = \oplus_\gamma \oplus_{kr} \oplus_i M(\mathcal{M}_{kr})^{(i)}$$

where γ runs among the automorphisms of curves, kr among the irreducible components of $\mathcal{M}_{g,[m]}(\gamma)$ and (i) follows a lambda-cotangent complex decomposition.

Further study should confirm **(1)** the role of $\mathrm{HH}_\bullet$ as a (weak) Tannaka fibre functor in $\mathrm{DA}^{et}(k)$, while **(2)** the $(\mathbb{S}^1, \mathbb{G}_m)$- (translation,filtration) for HH-spectra in $\mathcal{SH}^{et}(k)$ should reflect the stack limit Galois action between distinct stack inertia and divisorial strata of [CoMa14].

3. Grothendieck Section Conjecture and Homotopic Obstruction

For X smooth variety over a field k and the question of local-global obstruction to the existence of rational points (Topic 1), a conceptual breakthrough is given by Harpaz-Schlank's étale homotopy sets $X^\bullet(hk)$ and their adelic rational versions $X(\mathbb{A})^\bullet$ of homological and topological types $\bullet \in \{h; (h,n)\}$ (resp. $\bullet \in \{\mathbb{Z}h; (\mathbb{Z}h, n)\}$) – see [HaSc13].

$$\begin{tikzcd} X(k) \arrow[r, "\kappa"] \arrow[d] & X(hk)^\bullet \arrow[d, "loc"] \\ X(\mathbb{A}) \arrow[r] & \prod X(hk_v)^\bullet \end{tikzcd}$$

They altogether provide **(1)** an *unified overview of the classical fin/fin-ab, Brauer-Manin and étale Brauer-Manin descent obstructions*, and **(2)** *an ideal context for Grothendieck's section conjecture* that embraces the anabelian and abelian geometry at once – e.g. (Harari-Stix's Cor. 9.13 ibid.): If $X(\mathbb{A})^{fin} \neq \emptyset$, then (AGS) has a section.

3.1. **Homotopic Obstruction in Family.** Quick's refined construction in certain model category of Pro-space of $X(hk) = \pi_0(\{X_{\bar{k}}\}_{et}^{\wedge, hG_k})$ – as homotopy fixed points simplicial set see [Qui10]– gives access to a homotopy section reformulation where one can incorporate further constructions of classical geometry.

Let us consider a Friedlander's geometric fibration $X \to S$ of geometrically unibranch schemes, and assume being given some cuspical data C_S and C_s for the basis $S(hk)$ and each k-rational fibres $X_s(hk)$. Under those assumptions, Corwin and Schlank establish that – [CoSc20]:

> *Let $X \to S$ as above. If the the injective (resp. surjective) section conjecture holds the basis S and the fibres C_s, $s \in S$, endowed with cuspidal data, then it does so for X.*

3.2. $\mathbb{A}^1$**- and Motivic Versions.** Regarding Noether's problem (Topic 1; in its rational connectedness variant) and $\mathbb{A}^1$-geometry, we further refer to [AsOs19] §4.2 for Asok's approach via $\mathbb{A}^1$-connectedness. For Topic 2, we refer to a question of Toën regarding a motivic version of $X(hk)$, where one replaces $\{-\}_{et}$ (resp. G_k) by M (resp. a certain motivic Tannaka Galois group G_{MM}) in the homotopy fixed point construction above.

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Reporters: Séverin Philip & Densuke Shiraishi