Mathematisches Forschungsintitut Oberwolfach

Mini-Workshop: Arithmetic Geometry and Symmetries around Galois and Fundamental Groups

Abstracts

Arithmetic and Homotopy of Moduli Stacks of Curves

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Let $\mathcal{M}_{g,[m]}$ be the moduli stacks of genus g curves with m-unordered marked points, that we consider endowed with their complementary divisorial and stack stratifications. The former is a *stratification at infinity* and is given by the topological type (g', m') of curves in the Deligne-Mumford compactification of stable curves $\mathfrak{M}_{g,[m]}$, while the later is local and is given by the flat *stratification by the automorphisms of curves.*

As $\mathbb{Q}$-stacks, the moduli spaces accept some Geometric Galois Representations

$$\text{(GGR)} \qquad \rho_{\vec{s}}\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}[\pi_1^{et}(\mathcal{M}_{g,[m]} \otimes \bar{\mathbb{Q}}, \vec{s})]$$

where $\vec{s}\colon \bar{\mathbb{Q}}\{\{\mathbf{q}\}\} \to \mathcal{M}_{g,[m]}$ is a tangential base point associated to a chosen $\mathbb{Q}$-rational structure on $\mathcal{M}_{g,[m]}$ – tangential base points define at once some geometric base points for the fundamental group and some $\mathbb{Q}$-rational π_1-sections. They allow to bypass Falting's limitation on rational markings on curves, and also to benefit for the study of (GGR) from the rational Knudsen-Mumford lower dimensional (g', m')-embeddings in terms of limit Galois representations. The stack structure, via Hurwitz spaces, draw some connections between Geometric Galois Representations and the Regular Inverse Problem.

By providing accessible geometries that capture key arithmetic properties, the moduli stacks of curves are fundamental spaces in arithmetic geometry, in anabelian geometry – e.g. the unordered marked $\mathcal{M}_{0,m}$ are anabelian–, and in motivic theory – see the category of Mixed Tate motives.

We report on recent results on the stack arithmetic of these spaces, and on works in progress on the use of homotopical methods: how this leads to a motivic interpretation of these higher symmetries, and to a finer understanding of the operadic and arithmetic properties of the divisorial stratification.

1. Stack Arithmetic of Curves (joint with S. Maugeais.)

The Deligne-Mumford stack structure of $\mathcal{M}_{g,[m]}$ is recovered through the *inertia group sheaf* $I_{\mathcal{M},x}$, which for a given point $x\colon \mathrm{Spec}\,\mathbb{Q} \to \mathcal{M}_{g,[m]}$ geometrically identifies with the finite automorphism group $I_{\mathcal{M},\bar{x}} \simeq Aut_{\mathbb{C}}(C_{\bar{x}})$ of a Riemann surfaces represented by $\bar{x}$. By Noohi's uniformization Theorem, it follows that

$$I_{\mathcal{M},\bar{x}} \hookrightarrow \pi_1^{et}(\mathcal{M}_{g,[m]} \otimes \bar{\mathbb{Q}}, \vec{s}),$$

i.e. the automorphisms of curves form some local *ghost loops subgroups* of the étale fundamental group. This raises the question of *describing the* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-action of* (GGR) *on the stack inertia groups of* $\mathcal{M}_{g,[m]}$, which is indeed of local-vs-global nature.

Because they form the first non-trivial stack stratas, we consider in what follows the case of the cyclic stack inertia group $I_{\mathcal{M},\bar{x}} \simeq \langle\gamma\rangle$ that we study via their associated special loci.

1.1. **Special Loci, Irreducible Components of Hurwitz Spaces.** Let thus $\mathcal{M}_{g,[m]}(G)$ denote the special loci attached to a finite order group G: $\mathcal{M}_{g,[m]}(G)$ is the $\mathbb{Q}$-stacks of curves C/S that admit a faithful G-action $G \hookrightarrow Aut_S(C)$. In terms of Galois action, we notice that an irreducible components of $\mathcal{M}_{g,[m]}(G)$ is *a priori* defined over a number field K: this implies the stability of the $\pi_1^{et}(\mathcal{M}_{g,[m]} \otimes \bar{\mathbb{Q}})$-conjugacy classes of G under the action of the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$. In genus 0 and for G cyclic, one proves that every such component is of the form $\mathcal{M}_{0,[m]+k}$ – with m permuted points and k fixed points, $k \in \{0,1,2\}$ – thus defined over $\mathbb{Q}$. This implies that the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-action stabilizes the conjugacy classes of the cyclic stack inertia in genus* 0.

In higher genus, this raises the question of finding *arithmetic coarse invariants* of the irreducible components of $\mathcal{M}_{g,[m]}(\gamma)$. The identification of the normalization of the special loci $\widetilde{\mathcal{M}}_{g,[m]}(G) \simeq \mathcal{M}_{g,[m]}[G]/Aut(G)$ as quotient of the Hurwitz space of G-covers $\mathcal{M}_{g,[m]}[G]$ reduces this question to the characterization of S-families of G-cover, which draws a first connection with the geometry of Hurwitz spaces.

For $G = \langle\gamma\rangle$, an answer is provided in terms of étale cohomology with the definition on the geometric fibers of some branching datas **kr**, which allows to establish:

Theorem ([5] - Th. 4.3)**.** *The stack of* γ*-special loci admits a finite decomposition in irreducible components given by:*

$$\mathcal{M}_{g,[m]}(\gamma) = \coprod \mathcal{M}_{g,[m],\mathbf{kr}}(\gamma),$$

where $\mathcal{M}_{g,[m],\mathbf{kr}}(\gamma)$ *denotes the* $\mathbb{Q}$*-stack of curves inducing* γ*-covers with given branching datas* **kr**.

In a similar way to the Deligne-Mumford proof of the irreducibility of $\mathcal{M}_{g,[m]}$, this result relies on the existence of a Teichmüller space that parametrizes unmarked curves with given **kr**-datas. An arithmetic property of G-covers appears for the general case, to ensure that *the field of moduli* K *of certain* γ*-covers is indeed a field of definition* – see (Seq/Split)-condition of [8].

As in genus 0, this implies this time in every genus the conjugacy-stability of $I_{\mathcal{M}} = \langle\gamma\rangle$ under a certain local $\mathrm{Gal}(\bar{K}/K)$-action. The comparison with the global $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action of (GGR) and its complete description requires the use of the divisorial stratification.

1.2. **Inertial Limit Galois Actions, a Tate-like Action.** Let η denote the generic point of an irreducible component $\mathcal{M}_{g,[m],\mathbf{kr}}(G)$ of a special loci $\mathcal{M}_{g,[m]}(G)$, and let $\kappa(\eta)$ denote its residue field. After rigidification, one obtains an *inertial Galois action* $\rho^I_\eta\colon \mathrm{Gal}[\overline{\kappa}(\eta)/\kappa(\eta)] \to Aut(I_\eta)$ on the generic stack inertia group $I_\eta > G$ of the component. Since a tangential structure on $\mathcal{M}_{g,[m]}$ is a formal neighbourhood of a singular stable curve, the comparison of this local Galois action to the global (GGR) is provided by a specialization result for Deligne-Mumford stacks – see §3.2 and §4.2 of [6]:

> *For any component of cyclic special loci, there exist a K-point of $\mathcal{M}_{g,[m],\mathbf{kr}}(G)$ and a tangential base point $\vec{s}$ of $\mathcal{M}_{g,[m]}$, such that the* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-action $\rho_{\vec{s}}$ of* (GGR) *induces the inertial action ρ^I_η.*

For G cyclic, the local inertial Galois action can indeed be proven to be given by a certain extension of the field of definition K discussed in the previous section. From the $\mathbb{Q}$-definition of the cyclic irreducible components given by the Theorem above, one establishes more precisely:

Theorem ([5] – Th. 5.4 & [6] – Cor. 4.6, Th. 4.8)**.** *The* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-representation of* (GGR) *induces a χ-conjugacy* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-action on the cyclic stack inertia of $\mathcal{M}_{g,[m]}$. For $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$:*

$$\sigma.\gamma = h^{-1}_{\gamma,\sigma}.\gamma^{\chi_\sigma}.h_{\gamma,\sigma} \text{ where } h_{\gamma,\sigma} \in \pi_1^{et}(\mathcal{M}_{g,[m]} \otimes \bar{\mathbb{Q}}, \vec{s}). \tag{1}$$

This result surprisingly depends on the geometry of the Hurwitz components and of their stable compactification: if the **kr**-data corresponds to a class of G-covers whose G-isotropy groups span G, the result then follows from Fried's branch cycle argument; the general case requires a fine deformation argument of G-covers which allows to *compare G-stratas of $\mathfrak{M}_{g,[m],\mathbf{kr}}$ and $\mathfrak{M}_{g-1,[m]+2,\mathbf{kr'}}$ of different topological and ramification types.* This comparison requires the choice of tangential $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-actions which are compatible at the level of the stack inertia groups. We refer to this process under the term *inertial limit Galois action.*

1.3. **Towards the Stack Arithmetic of Higher Stratas.** By analogy with the divisorial arithmetic of $\mathcal{M}_{g,[m]}$, this Tate-like action motivates further studies of the higher arithmetic of the stack stratification. The inertial limit Galois action provides for example a mean of comparing the conjugacy factors of the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action between *stratas of different topological types.*

We mention two immediate directions of research:

(i) *determines some* discrete arithmetic invariants *of the irreducible components of $\mathcal{M}_{g,[m]}(G)$ for non-abelian groups G;*

(ii) *complete the description of the* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-action Eq.* (1) *by determining* relations for the conjugacy factors.

As shown in the case of the cyclic strata, progress will certainly rely on a fine understanding of the arithmetic of G-covers and of the Hurwitz spaces. As more concrete examples, let us mention for (i) the use in [7] of H^2-data in terms of mixed cohomology that complete the monodromy invariants for cyclic extensions.

For (ii), relations could come either from the comparison of different stack inertia groups and follow either from (i), or from the use of the topological type: equations ($\star$)-($\star\star$) of [21], and their generalization (R) in [3], are some examples of such comparisons – in these cases of $\mathcal{M}_{0,[5]}(\mathbb{Z}/2\mathbb{Z})$ with respect to $\mathcal{M}_{0,[4]}$, of $\mathcal{M}_{0,[6]}(\mathbb{Z}/3\mathbb{Z})$ with respect to $\mathcal{M}_{0,[4]}$, and of $\mathcal{M}_{1,[2]}(\mathbb{Z}/2\mathbb{Z})$ with respect to $\mathcal{M}_{1,1}$ –, see also [19].

In another direction, and always by analogy with the Galois divisorial arithmetic, the Tate-like action of Eq. (1) raises the question of a motivic interpretation of this result.

2. Motivic Stack Considerations

Let k be a number field, and let $MT(k)$ denote the category of Mixed Tate motives over k. This is a Tannakian category of group G_{MT}, neutralized by the canonical Adams weight fibre functor, whose properties are tightly related to the divisorial arithmetic of the moduli schemes $\mathcal{M}_{0,m}$: it is motivically generated by the $\mathcal{M}_{0,m}$s [2], a $p+2q$-motivic weight comes from a p-codimensional component of $\mathfrak{M}_{0,m}$ and of a q-Tate twist, relation between periods are induced by the Knudsen morphisms [22].

Motivated by the Tate-like $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action on the cyclic stack inertia, we present how Morel-Voedsky's motivic homotopy provides a convenient context where to *develop motivic properties for the stack inertia that are Tate-compatible.* We present the main difficulties, illustrate the approach in the case of representable, and generally refer to the forthcoming [4] for the case of stacks.

2.1. Motivic Homotopy Theory and Stacks.

The motivic context is given by the unstable-stable motivic categories $\mathcal{H}(\mathbb{Q}) \rightleftarrows \mathcal{SH}(\mathbb{Q})$, respectively defined as the homotopy categories of spaces $Sp(\mathbb{Q}) = sPr(Sm_{\mathbb{Q}})$ and $\mathbb{P}^1$-spectras over $Sp(\mathbb{Q})$ endowed with their $\mathbb{A}^1$-local injective model category with respect to the étale topology. The category $\mathcal{SH}(\mathbb{Q})$ is triangulated, equivalent to Voevodsky's $DM(\mathbb{Q})$, and has the Lefschetz motive inverted at the level of morphisms as a result of the $\Sigma_{\mathbb{P}}$-stabilization.

A first connection between the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-stack inertia framework and the $\mathcal{H}(\mathbb{Q})$ context is given by replacing the Galois category formalism of SGA1 by Artin-Mazur and Friedlander [12] simplicial étale topological type one: we attache to the stack $\mathcal{M}_{g,[m]}$ – and similarly to its stack inertia $I_{\mathcal{M}}$ – first a pro-space $\{\mathcal{M}_{g,[m]}\}_{et}$, then by Isaksen's étale realization functor [16] a $\mathbb{A}^1$-type $\{\mathcal{M}_{g,[m]}\}_{\mathbb{A}^1}$ in $sPr(Sm_{\mathbb{Q}})$.

A more intrinsic context is indeed given by considering $\mathcal{M}_{g,[m]}$ as a specific object of $sPr(Sm_{\mathbb{Q}})$, whose Giraud's descent property is characterized in terms of hypercovers [10]: within this context, the (group) stack inertia identifies to a derived loop space $I_{\mathcal{M}} \simeq RHom(\mathbb{S}^1, \mathcal{M})$ (resp. to a homotopy group sheaf).

This approach places in particular the stack motivic study of $\mathcal{M}_{g,[m]}$ within Toën-Vezzosi's Homotopical Algebraic Geometry theory (HAG) [24]. Since $\mathbb{P}^1 \simeq$

$\mathbb{S}^1 \wedge \mathbb{G}_m$ within $\mathcal{H}(\mathbb{Q})$, and since $\mathbb{G}_m$ is the Lefschetz motivic divisorial monodromy of $\mathcal{M}_{g,[m]}$, one concludes that:

> *the motivic homotopy theory of $\mathcal{M}_{g,[m]}$ gives a favourable context where to illustrate how the $\mathbb{S}^1$-loops encode the ghost 2-structure of motivic spectras.*

2.2. **Mixed Tate Motives and Beyond.** By contrast, we now illustrate the relevance and the non-triviality of this approach on the specific examples of representable stacks $\mathcal{M}_{0,m} \in Sm_{\mathbb{Q}}$. On one side, the homotopical Mixed Tate framework is given by [17] and follows Spitzweck's representation theorem for cells modules over an Adams graded cycles algebra, as provided by Bloch-Kriz's $\mathcal{N}_{BK}$ – see op.cit.

On the other side, the HAG context is provided by Toën's Spec-functor of [23] Spec : $Alg_{\mathbb{Q}}^{\triangle^o} \to sPr(\mathbb{Q})$, and by Hitchin's Quillen equivalence $Alg_{\mathbb{Q}}^{\triangle^o} \rightleftarrows cdga_{\mathbb{Q}}$. As a result, since the Bar complex is an homotopy colimit of diagrams, one obtains that (the prounipotent part of) G_{MT} *is weakly equivalent to the derived loop space of Spec*$\mathcal{N}_{BK}$. Since the prounipotency of the homotopy group sheaf characterizes the schematic image in $sPr(Sch_Q)$, notice that a similar construction for $\mathcal{M}_{g,[m]}$ that realizes $I_{\mathcal{M}}$ as motivic object requires to enlarge the aforementioned Quillen equivalence. This lead to the DAG-context that allows to capture the $\mathbb{S}^1$-motivic inertia.

Despite the relevance of this approach, a final and fundamental difficulty is still given by the *question of a neutralizing fibre functor*, which must induces *non-2-trivial* geometrical de Rham-Betti comparison isomorphisms: as an étale-locally quotient stack, the rational cohomology of $\mathcal{M}_{g,[m]}$ is whose of its coarse scheme $M_{g,[m]}$. An answer is here again provided by the HAG context, that gives computable *stack inertia periods* in terms of iterated integrals that are compatible with the Tannakian weight – via $\mathcal{M}_{0,m} \to \mathcal{M}_{0,[m]}$ and the choice of tangential structures as involved for Galois and $\mathcal{M}_{g,[m],\mathbf{kr}}(\gamma)$ see [4].

3. Arithmetic of Operads

We present how the divisorial stratification of $\mathcal{M}_{g,[m]}$ supports some fundamental arithmetic and geometric properties. The question is two fold: from the geometric point of view, it is related to the fundamental question of defining *on smooth objects an operadic structure that is given by singular degeneracies*; from the arithmetic point of view it is related to Grothendieck-Teichmüller theory that is to determine *how* Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$ *is encoded within the geometric symmetries of* $\mathcal{M}_{g,[m]}$. More precisely, GT theory provides a finitely presented group $\widehat{GT}$ that contains Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$ and factorizes the (GGR) [9, 14].

We report on how the homotopy theory of spaces and the notion of tangential structures on $\mathcal{M}_{0,m}$ provides some insight on these questions. We were recently informed that K. Wickelgren and C. Westerland developed an independent and similar approach in the case of configuration spaces – see *K. Wickelgren, Operad Structure on $Conf_n$* in this volume.

3.1. **Genus Zero Moduli Spaces.** Motivation for this work comes from the recent operadic result of B. Fresse and G. Horel [11, 13], that interprets Drinfel'd definition of $\widehat{GT}$ in terms of operad in prospaces.

Theorem (Fresse, Horel)**.** *The group $\widehat{GT}$ is isomorphic to the homotopy group of (pro) little 2-discs operads $E_2^\wedge$.*

Here $E_2^\wedge$ denotes either the Sullivan rational model or the completion in prospaces. Their fundamental groups are respectively given by the Malčev and the profinite completion of the parenthesized braid operad in groupoids $PaB^\wedge$. Since the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is contained in $\widehat{GT}$, this induces a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action on $PaB^\wedge$ that is group-theoretically defined and from topological origin.

We deal with the question of recovering this result – more precisely the refinement of [1] – in terms of an arithmetic $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action at the level of a $\mathbb{Q}$-operadic structure on $\mathcal{M}_{0,m}$ then $\mathcal{M}_{0,[m]}$. To fix some (GGR) or $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-actions requires to specify some $\mathbb{Q}$-tangential structures on $\mathcal{M}_{0,m}$, i.e. to choose a formal neighbourhood $\mathrm{Spec}\,\mathbb{Q}[[\mathbf{q}]] \to \mathfrak{M}_{0,m}$ of some singular curves in the Deligne-Mumford compactification of $\mathcal{M}_{0,m}$ [15, 6].

In terms of operads, the choice of a tangential structure on the spaces defines the geometric operadic composition morphisms and ensures that they are $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant. We obtain a refinement of the Theorem above, that is from arithmetic origin.

Theorem (conjecture)**.** *The set of $\mathbb{Q}$-tangential structures $\{\vec{s}\}$ over $\mathcal{M}_{0,m}$ defines an operad $\mathcal{M}_{\mathbb{Q}} = \{(\mathcal{M}_{0,m}, \vec{s})\}_{m,\vec{s}}$ in $\mathbb{Q}$-Proschemes whose geometric étale homotopy type is endowed with a* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*-action – see Eq.* (GGR)*.*

The approach is based on Grothendieck's formal-algebraization deformation theory for curves; the operad $\mathcal{M}_{\mathbb{Q}}$ is defined in terms of Friedlander Artin-Mazur étale topological type. As a result, we obtain after completion an operad in prospaces $\mathcal{M}_{\mathbb{Q}}(\bar{\mathbb{Q}})^\wedge$ which encodes some arithmetic a priori not distinguished by $\widehat{GT}$. More precisely, $\mathcal{M}_{\mathbb{Q}}(\bar{\mathbb{Q}})^\wedge$ is weakly equivalent to the framed little 2-discs operads $FE_2^\wedge$, while $\widehat{GT} \simeq Aut^h(E_2^\wedge)$ and $Aut^h(E_2^\wedge) \simeq Aut^h(FE_2^\wedge)$.

This property can already be seen at the level of configuration spaces and braids groups – PaB is a model for E_2 –, by providing a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action on an extension of $PaB^\wedge$ that descends to the classical $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action of Ihara-Matsumoto on Braids groups [14].

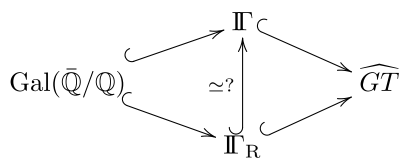
Following Mac Lane's coherence Theorem, this approach provides in particular a computable $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action which is entirely defined in 1, 2 and 3-arity. Let us mention that Serre's anabelian bonté of $\mathcal{M}_{0,m}$ plays a key role in defining the operadic composition on $\mathcal{M}_{\mathbb{Q}}$.

3.2. **Towards Higher Symmetries.** Because this approach is close to the geometry of curves and already provides some refinement in genus zero, it motivates and gives access to further developments in the direction of *stack and higher genus symmetries.*

In terms of stack, this motivates our work in progress on defining a similar rational operad for the genus zero moduli stack of curves with unordered marked points $\mathcal{M}_{0,[m]}$. The ∞-model category of [20] provides the necessary context to connect the tangential arithmetic and the 2-structure of $\mathcal{M}_{0,[m]}$. In this case, the homotopy groups are *not torsion-free* – unlike the braid groups in the previous case – but contains some stack torsion like $\pi_1^{orb}(\mathcal{M}_{0,[m]}(\mathbb{C}))$ does.

Regarding the moduli spaces in higher genus, the operad $\mathcal{M}_{\mathbb{Q}}$ already comes with additional commutativity- and associativity-like constraints at the level of braided monoidal category. This provides additional GT-like equations, which while already included in the pentagon-hexagon equations I, II and III defining $\widehat{GT}$, motivates in higher genus the study of a potential refinement of the original group $\mathbb{I}\Gamma$ of [18].

(3.2.1) Define in higher genus a Grothendieck Teichmüller group $\mathbb{I}\Gamma_{\mathrm{R}}$ *given by relations based on the tangential refined associativity and commutativity constraints:*



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