ACTION OF A GROTHENDIECK-TEICHMÜLLER GROUP ON TORSION ELEMENTS OF FULL TEICHMÜLLER MODULAR GROUPS OF GENUS ONE

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ABSTRACT. In this paper we define a new version of Grothendieck-Teichmüller group $\widehat{GR}$ defined by three generalized equations coming from finite order diffeomorphisms, and we prove that it is isomorphic to the known version $\mathbb{I}\Gamma$ of the Grothendieck-Teichmüller defined in [17]. We show that $\widehat{GR}$ acts on the full mapping class groups $\pi_1^{geom}(\mathcal{M}_{g,[n]})$ for $2g-2+n>0$. We then prove that the conjugacy classes of prime order torsion of $\pi_1^{geom}(\mathcal{M}_{1,[n]})$ are exactly the *discrete* prime order ones of $\pi_1^{orb}(\mathcal{M}_{1,[n]})$. Using this we prove that $\widehat{GR}$ acts on prime order torsion elements of $\pi_1^{geom}(\mathcal{M}_{1,[n]})$ in a particular way called λ-conjugacy, analogous to the Galois action on inertia.

KEYWORDS: moduli spaces of curves; mapping class groups; elliptic curves.

MATHEMATICS SUBJECT CLASSIFICATION 2000: 11F06, 11G30, 14H45, 14G32

1. INTRODUCTION

1.1. **Grothendieck-Teichmüller action on $\widehat{\Gamma}_{0,[n]}$ and $\widehat{\Gamma}_{g,n}$.** The group $\widehat{GT}$, introduced by V. G. Drinfel'd in [4] is defined in the framework of genus zero moduli spaces, as the group of the invertible elements $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{\mathbb{F}}_2'$ – where *invertible* is explained below – and which satisfy the equations

$$\text{(I)} \qquad f(x,y)f(y,x) = 1$$

$$\text{(II)} \qquad f(x,y)x^m f(z,x)z^m f(y,z)y^m = 1$$

$$\text{(III)} \qquad f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51})f(x_{12},x_{23}) = 1$$

where $xyz = 1$, $m = (\lambda-1)/2$. It contains the absolute Galois group and comes with an explicit action on the full genus zero mapping class groups

$$\sigma_i \mapsto f(y_i,\sigma_i^2)\sigma_i^\lambda f(\sigma_i^2,y_i) \qquad (\lambda,f) \in \widehat{GT}$$

where $y_i = \sigma_{i-1}\sigma_{i-2}\cdots\sigma_1^2\cdots\sigma_{i-2}\sigma_{i-1}$ and σ_i are generators of $\widehat{\Gamma}_{0,[n]}$. The pair (λ, f) is *invertible* as an automorphism of $\widehat{\Gamma}_{0,4}$.

It is not known whether $\widehat{GT}$ acts on the profinite mapping class groups of all genera. However, there exists a subgroup $\mathbb{I}\Gamma \subset \widehat{GT}$ which has been shown to act on all *pure* mapping class groups $\widehat{\Gamma}_{g,n}$ [5, 17].

Definition 1.1. *The group* $\mathbb{I}\Gamma$ *is the subgroup of* $\widehat{GT}$ *composed by elements* $(\lambda, f) \in \widehat{GT}$ *which satisfy the two equations*

$$\text{(III')} \qquad f(\sigma_1\sigma_3, \sigma_2^2) = g(x_{45}, x_{51}) f(x_{12}, x_{23}) f(x_{34}, x_{45})$$

$$\text{(IV)} \qquad f(\sigma_1, \sigma_2^4) = \sigma_2^{\rho_2(F)} f(\sigma_1^2, \sigma_2^2) \sigma_1^{4\rho_2(F)} (\sigma_1\sigma_2)^{-6\rho_2(F)}$$

where $g \in \widehat{\mathbb{F}}_2$ *is the single element such that* $f(x,y) = g(y,x)^{-1} g(x,y)$, $\rho_2 : \widehat{GT} \to \widehat{\mathbb{Z}}$ *is defined by* $g \equiv (xy)^{\rho_2(F)}$ *in* $\widehat{\mathbb{F}}_2^{ab}$ *in* [7]*, and* σ_i *denotes the i-th standard generator of profinite Artin braid group* $\widehat{B}_n$*.*

An action of $\mathbb{I}\Gamma$ on $\widehat{\Gamma}_{g,n}$ is given for each choice of quilt-decomposition on a surface of type (g, n) – geometrically this corresponds to a choice of tangential base point at infinity on the moduli space $\mathcal{M}_{g,n}$. These actions all differ by inner automorphisms.

On the other hand, the choice of a tangential base point gives rise to a $G_{\mathbb{Q}}$-action on pure mapping class groups. In [17] H. Nakamura and L. Schneps show that there exists a tangential base point such that the $\mathbb{I}\Gamma$-action extends to pure profinite mapping class groups with boundary, and extends the $G_{\mathbb{Q}}$-action on these groups with respect to that base point.

Theorem 1.2 ([17])**.** *Let* $\Gamma_{g,n}^b$ *be the pure mapping class group of a surface of type* (g,n) *with b boundaries equipped with a quilts-pants decomposition* Q/P*. Then there exists a homomorphism*

$$\psi_{Q/P} \colon \mathbb{I}\Gamma \to Aut(\widehat{\Gamma}_{g,n}^b).$$

One of the goals of this paper is to show that the action of $\mathbb{I}\Gamma$ on pure mapping class groups in fact extends to the full mapping class groups – see theorem 2.2:

Theorem A. *The Grothendieck-Teichmüller group* $\mathbb{I}\Gamma$ *acts on the full mapping class groups* $\widehat{\Gamma}_{g,[n]}$*.*

We deduce this result by studying the usual actions defined on pure and braid generators for a Wajnryb-type presentation of the full mapping class group.

The second result of this paper is the comparison of $\mathbb{I}\Gamma$ with another version of the Grothendieck-Teichmüller group $\widehat{GR}$ defined by a torsion-type relation. This relation is related with $\widehat{GS}_{0,0}^1$ defined by morphisms in genus zero – see [20]. We prove that $\widehat{GR}$ is isomorphic to $\mathbb{I}\Gamma$; thus $\mathbb{I}\Gamma$ also comes from respection torsion type morphism. This group contains the absolute Galois group.

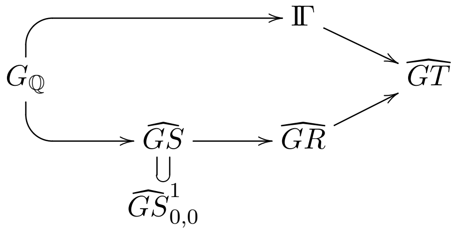
We next prove that profinite torsion of prime order in genus one is conjugate to discrete torsion, and finally that $\mathbb{I}\Gamma$-action on such torsion is given by λ-conjugacy. These extend similar results in genus zero [3].

1.2. Torsion and Grothendieck-Teichmüller groups.

The defining relations of $\widehat{GT}$ arise from requiring that the $\widehat{GT}$ action respect certain morphisms between moduli spaces of dimension one and two, for example point erasing morphisms in genus zero – see [11]. In the same manner, one can consider relations arising from respecting morphisms associated to quotient surfaces by a finite order diffeomorphisms – i.e. torsion elements of mapping class groups. In genus zero, such torsion-type relations between $\Gamma_{0,[6]}$, $\Gamma_{0,[5]}$ and $\Gamma_{0,[4]}$ define a Grothendieck-Teichmüller group $\widehat{GS}_{0,0}^1$.

Unlike $\widehat{GT}$ which is known to contain the absolute Galois group $G_{\mathbb{Q}}$, the group $\widehat{GS}^1_{0,0}$ only contains the absolute Galois group of $\mathbb{Q}^{ab}(\{\sqrt[n]{2}\}_{n\geqslant 1})$. This group $\widehat{GS}^1_{0,0}$ was generalized by H. Tsunogai [23] to a group $\widehat{GS}$ which contains $G_{\mathbb{Q}}$.

In section 3 we define a group $\widehat{GR}$, related to $\widehat{GS}^1_{0,0}$ in the sense that it also comes from respecting torsion-type relations, and has the properties below:



By Grothendieck-Teichmüller computation we prove:

Theorem B. *The groups $\widehat{GR}$ and $\mathbb{I}\Gamma$ are isomorphic.*

Thus by theorem A, $\widehat{GR}$ acts on the full mapping class groups – remark however that $\widehat{GR}$ is defined from torsion conditions and $\mathbb{I}\Gamma$ is not.

1.3. **Profinite torsion in genus one.** Let us recall the following isomorphisms between orbifold, resp. stack, geometric fundamental groups and discrete, resp. profinite, mapping class groups

$$\pi_1^{orb}(\mathcal{M}_{1,[n]}) \simeq \Gamma_{1,[n]} \quad \text{and} \quad \pi_1^{geom}(\mathcal{M}_{1,[n]}) \simeq \widehat{\Gamma}_{1,[n]}.$$

Mapping class groups are residually finite groups, i.e. they inject into their profinite completions

$$\Gamma_{1,[n]} \hookrightarrow \widehat{\Gamma}_{1,[n]},$$

and thus torsion elements of the discrete orbifold group keep the same order in the geometric algebraic groups $\pi_1^{geom}(\mathcal{M}_{1,[n]})$. It is then a result of S. P. Kerckhoff [9] that these elements correspond exactly to geometric automorphism of curves, i.e. to *inertia groups of stacks* $I_x \subset \pi_1(\mathcal{M}_{1,[n]})$.

Following [13] - questions 3.6 and 3.5, one can ask *if all profinite torsion elements are conjugate to discrete ones* and *to describe a Grothendieck-Teichmüller action on such torsion.* In genus one, we give a positive but partial answer to the former:

Theorem C. *All prime order torsion elements of $\widehat{\Gamma}_{1,[n]}$ are conjugate to discrete ones.*

Finally we use this to show that the Grothendieck-Teichmüller action is given by *λ-conjugacy*, which means that an element $(\lambda, f) \in \widehat{GR}$ acts on a prime order torsion element γ by conjugating it and raising it to the power λ. Let us recall that this is a typical feature of Galois action on inertia.

Theorem D. *An element (λ, f) of the Grothendieck-Teichmüller group $\widehat{GR}$ acts on the prime order torsion elements of the full profinite mapping class group $\widehat{\Gamma}_{1,[n]}$ by λ-conjugacy.*

The approach taken to establish Theorem C is *Serre theory* for good groups already developed in [3] for genus zero, where analogous results to Theorems C and D are established.

2. A $\mathbb{I}\Gamma$-action on $\widehat{\Gamma}_{g,[n]}$

We establish that the Grothendieck-Teichmüller group $\mathbb{I}\Gamma$ admits an action on *full mapping class groups* that extends the known action on pure groups.

We first need to choose a convenient finite presentation of $\Gamma_{g,[n]}$. Let α_i, β_i, γ denote the Dehn twists and σ_i the braid twists as in Fig. 1 below.

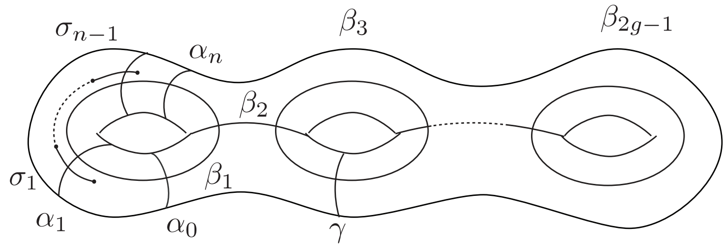


FIGURE 1. Generators of $\Gamma_{g,[n]}$

Let $S_{g,n}$ denote a topological surface of genus g and n boundary components or marked points. We say $S_{g',n'}$ is a piece of $S_{g,n}$ if it is cut out of $S_{g,n}$ by a set of simple closed curves.

Proposition 2.1 ([10])**.** *Let $g \geqslant 1,\ n \geqslant 0$ and let*

$$\mathcal{G}_{g,n} = \{\alpha_0, \alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_{2g-1}, \gamma; \{\sigma_i\}_{1\leqslant i\leqslant n-1}\}.$$

Let us consider the Wajnryb presentation of $\Gamma_{g,n}$, with relations denoted by R. Then a presentation of the full mapping class group $\Gamma_{g,[n]}$ *is given by $\mathcal{G}_{g,n}$ as a system of generators with relations below:*

- (B) *Braids: for two braid-twists $\sigma_i\sigma_j = \sigma_j\sigma_i$ if supports are non-intersecting, $\sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i$ if supports intersect transversally in one point.*
- (L) *Lanterns: between a twist α_i and a braid-twist σ_i intersecting transversally on a piece of type $S_{0,4}$*

$$\sigma_i\alpha_i\sigma_i\alpha_i = \alpha_{i-1}\alpha_{i+1}.$$

- (P) *Pure relations R.*
- (C) *Non-explicit relations expressing σ_i^2 in terms of α_i, β_i and γ.*

Proof. It is known that the presentation of $\Gamma_{g,n}$ of Wajnryb [24] uses the α_i, β_i and γ of $\mathcal{G}_{g,n}$, so $\mathcal{G}_{g,n}$ is a generating set. For the full group, we follow a general result of [8] on group presentations given by extension, applied in our case to the permutation exact sequence:

$$1 \longrightarrow \Gamma_{g,n} \longrightarrow \Gamma_{g,[n]} \longrightarrow \mathfrak{S}_n \longrightarrow 1. \quad (1)$$

According to this, a system of generators of $\Gamma_{g,[n]}$ is given by the union of a system of generators of $\Gamma_{g,n}$ with a set of the preimages of generators $\mathfrak{S}_n$ – here the braid-twists σ_i. Hence $\mathcal{G}_{g,n}$ is a generating set of $\Gamma_{g,[n]}$.

Relations are then of three kinds: those given by relations (P) in the presentation of $\Gamma_{g,n}$, relations (B) and (C) given by lifting relations in the quotient group $\mathfrak{S}_n$, and relations (L) of the kind

$$hkh^{-1} = w \tag{2}$$

where h is a non-pure element of $\mathcal{G}_{g,n}$, k is a pure element of $\mathcal{G}_{g,n}$ and w is element of $\Gamma_{g,n}$. Relations of type (B) and (C) are the braid relations of the statement, because they are precisely the liftings of the usual presentation of S_n by transpositions.

Relations (L) are of the form $\sigma_i\beta_j\sigma_i^{-1} = \beta_j$, $\sigma_i\gamma\sigma_i^{-1} = \gamma$, $\sigma_i\alpha_j\sigma_i^{-1} = \alpha_j$ for $j \neq i$, which are trivial as the Dehn twists commutes, and $\sigma_i\alpha_i\sigma_i^{-1} \in \Gamma_{g,n}$ for $1 \leqslant i \leqslant n-1$. The topological piece supporting the underlying curves s_i, a_i of σ_i, α_i is of type $S_{0,4}$, bounded by two marked points and curves a_{i-1} et a_{i+1}. Since $\sigma_i\alpha_i\sigma_i^{-1}$ is the Dehn twist along the curve $\sigma(a_i)$, one identifies a Johnson lantern and relations take the form

$$\sigma_i^2 \cdot \alpha_i \cdot \sigma_i\alpha_i\sigma_i^{-1} = \alpha_{i-1}\alpha_{i+1} \tag{3}$$

Since α_{i-1} α_{i+1} commute with σ_i one obtains the lantern relation (L)

$$\sigma_i\alpha_i\sigma_i\alpha_i = \alpha_{i-1}\alpha_{i+1}.$$

□

Recall that following [17] an action of $\mathbb{I}\Gamma$ is defined on $\widehat{\Gamma}_{g,n}$ once a pants decomposition P and an associated quilts decomposition P/Q of the surface $S_{g,n}$ is fixed. The action of $\mathbb{I}\Gamma$ on any twist is then computed using A/S-moves between curves of the pants and support of the twist – we refer to [17] for details on this *Lego*:

$$\begin{aligned}
&\psi_{Q/P}(F)\colon \alpha \to \alpha^{\lambda} \text{ if } a \in P\\
&\psi_{Q/P}(F)\colon \beta \to \mathrm{Inn}[f(\beta,\alpha)](\beta^{\lambda}) \text{ if } b \to a \text{ is an } A\text{-move.}\\
&\psi_{Q/P}(F)\colon \gamma \to \mathrm{Inn}[\gamma^{-8\rho_2} f(\gamma^2,\alpha^2)\alpha^{8\rho_2}(\gamma\alpha\gamma)^2](\gamma^{\lambda}) \text{ if } c \to a \text{ is an } S\text{-move.}
\end{aligned} \tag{4}$$

where $\mathrm{Inn}(g)(x) = gxg^{-1}$ and roman letters denote the supports of the associated Dehn twists. The final action is then given by move compositions:

$$\psi_{Q/P}(F)\colon \alpha \to \mathrm{Inn}[\prod_{i=1}^{n} f(P_i \overset{A/S}{\dashrightarrow} P_{i+1})](\alpha^{\lambda})$$

We now fix the pants-quilts decomposition as in Fig. 2.

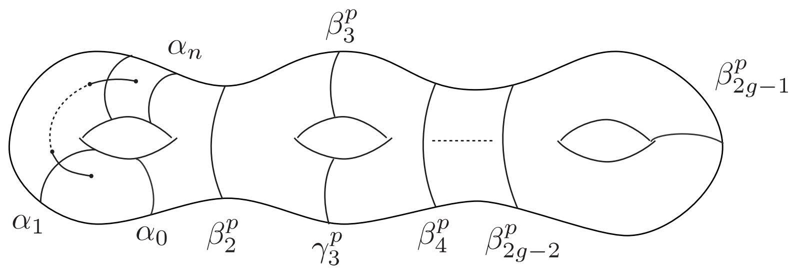


FIGURE 2. Quilts-Pants decomposition of $S_{g,n}$

Theorem 2.2. *Let Q/P be the quilts-pants decomposition of $S_{g,n}$ in Fig. 2. Then it defines an action of* $\mathbb{I}\Gamma$ *on* $\widehat{\Gamma}_{g,[n]}$

$$\psi_{P/Q}: \mathbb{I}\Gamma \to Aut_P(\widehat{\Gamma}_{g,[n]})$$

given on σ_i by:

$$F(\sigma_i) = f(\sigma_i^2, \alpha_i)\sigma_i^\lambda f(\alpha_i, \sigma_i^2) \quad 1 \leqslant i \leqslant n.$$

This action preserves the symmetric group.

The proof of this theorem underlines the key role played by subsurfaces of type $(g, n) = (0, 4)$ in pants-quilts decompositions. Let us first recall the following technical lemma – established in [17, Lemma 1.5].

Lemma 2.3. *Let x, y, z be elements of $\Gamma_{g,[n]}$ such that $xyz = c$ commutes with each element x, y and z. Then for all $F \in \mathbb{I}\Gamma$*

$$f(x,y)x^m f(z,x)z^m f(y,z)y^m = c^m \text{ where } m = (\lambda - 1)/2 \tag{5}$$

$$f(x,y)x^{\tilde{m}} f(z,x)z^{\tilde{m}} f(y,z)y^{\tilde{m}} = c^{\tilde{m}} \text{ where } \tilde{m} = -(m+1). \tag{6}$$

We can now proceed to the proof of theorem 2.2.

Proof of theorem 2.2. The action is defined on the pure system generators of $\mathcal{G}_{g,n}$ of proposition 2.1 after composition of A/S-moves starting from a Q/P quilts-pants decomposition. Following the expression on the σ_i, one observes that the permutation associated to each σ_i is preserved, proving the last statement of the theorem.

Let us show that such action is well-defined by preserving relations of proposition 2.1. Relations (P) of $\Gamma_{g,n}$ are preserved following theorems of [17]. Then the use of the pentagon relation (III) shows that the $\mathbb{I}\Gamma$-action $\psi_{P/Q}$ on $\widehat{\Gamma}_{g[n]}$ is conjugate to the original action of [4] by

$$f(\alpha_{n-1}, y_n) f(\alpha_{n-2}, y_{n-1}) \cdots f(\alpha_2, y_3) f(\alpha_1, y_2).$$

Braid-relations (B) between σ_i are then preserved since they are by the Drinfel'd action.

Consider lantern relation (L) between $\delta = \alpha_{i-1}$, $\varepsilon = \alpha_{i+1}$, $x = \sigma_i^2$, $y = \alpha_i$ and $z = \sigma_i \alpha_i \sigma_i^{-1}$ where α_i and α_{i+1} are boundary components $xyz = \delta\varepsilon$.

Recall that as supports of δ, ϵ and y belong to the pants decomposition P, they are raised to λ power. Moreover, curves associated to x and y differ by an A-move on a lantern, hence

$$F(x) = f(x,y)x^{\lambda} f(y,x).$$

Let us compute $F(z) = F(\sigma_i)F(y)F(\sigma_i^{-1})$ term-by-term, where by definition of the action

$$F(\sigma_i) = f(x,y)\sigma_i^{\lambda} f(y,x).$$

As $f \in \widehat{\mathbb{F}}'_2$, one has

$$\begin{aligned} f(y,x) &= f(y,\sigma_i^2) \\ &= \sigma_i^{-1} f(\sigma_i y \sigma_i^{-1}, \sigma_i^2)\sigma_i \\ &= \sigma_i^{-1} f(z,x)\sigma_i \end{aligned}$$

which by substituting in the last term gives

$$F(\sigma_i) = f(x,y)x^m f(z,x)\sigma_i.$$

Since $xyz = \delta\varepsilon$ commutes with x, y and z, one applies relation (5) of lemma 2.3 to obtain

$$F(\sigma_i) = y^{-m} f(z,y) z^{-m}\sigma_i y^{-m}(\delta\varepsilon)^m$$

where $z^{-m}\sigma_i = \sigma_i y^{-m}$, hence the final expression of $F(z)$:

$$F(z) = y^{-m} f(z,y) z^{\lambda} f(y,z) y^m.$$

The action on the lantern $xyz = \delta\varepsilon$ leads to

$$f(x,y)x^{\lambda} f(y,x) y^{m+1} f(z,y) z^{\lambda} f(y,z) y^m = (\delta\varepsilon)^{\lambda},$$

and finally, writing $x^{\lambda} = x^m x^{m+1}$ and $z^{\lambda} = z^{m+1}z^m$, and using the fact that δ and ε commute with x, y, z, this is equivalent to

$$x^{m+1} f(y,x) y^{m+1} f(z,y) z^{m+1} \cdot z^m f(y,z) y^m f(x,y) x^m = \delta^{\lambda}\epsilon^{\lambda}.$$

This equality then holds thanks to relations (5) and (6) of lemma 2.3 which concludes the proof.

Finally, to show that the $\mathbb{I}\Gamma$-action respects the relations (C) expressing σ_i^2 in terms of α_i, β_i and γ, it is not necessary to know these relations explicitly, thanks to the fact that the $\mathbb{I}\Gamma$-action on $\widehat{\Gamma}_{g,n}$ satisfies the Lego property (4). Indeed σ_i^2 is a Dehn twist along a loop obtained from the loop supporting α_i and therefore $F(\sigma_i^2) = f(\sigma_i^2, \alpha_i)\sigma_i^{\lambda} f(\alpha_i, \sigma_i^2)$ as desired. □

Remark 2.4. It can be computed that the action of $\mathbb{I}\Gamma$ on generators $\mathcal{G}_{g,n}$ as previously defined is given by λ-conjugacy on each generator. We do not reproduce this computation here as it is not of any help for the previous proof.

Corollary 2.5. *Let Q/P be the quilts-pants decomposition of $S_{g,n}$ in Fig. 2. Then the action of $\mathbb{I}\Gamma$ on $\widehat{\Gamma}_{g,[n]}$ of the previous theorem defines an action:*

$$\psi_{P_0/Q}\colon \mathbb{I}\Gamma \to Aut_{P_0}(\widehat{\Gamma}^2_{0,[n]})$$

where P_0 is the quilt decomposition induced by Q/P on $S^2_{0,[n]}$. This action $\psi_{P_0/Q}$ is conjugate to the usual action of $\widehat{GT}$ on $\widehat{\Gamma}_{0,[n]}$.

Proof. Let us cut the surface $S_{g,n}$ along the curves α_0 and α_n, thus giving two surfaces with boundaries $S_{0,n}^2$ and S_{g-1}^2. The remaining of the quilt-pants decomposition Q/P gives a pants decomposition on each part, and the expression of the action of $\mathbb{I}\Gamma$ on $\widehat{\Gamma}_{g,[n]}$ of theorem 2.2 defines a $\mathbb{I}\Gamma$ action on $\widehat{\Gamma}_{0,[n]}^2$.

This action is conjugate to the usual $\widehat{GT}$ action and the conjugacy factor is given by composition of A/S-moves from the pants decomposition P_0 to the usual pants decomposition of $S_{0,n}$. □

3. The $\widehat{GR}$ Grothendieck-Teichmüller Group

In this section we define $\widehat{GR}$ a new version of the Grothendieck-Teichmüller group having the following properties: it contains the absolute Galois group $G_{\mathbb{Q}}$, acts on all full mapping class groups and it is defined by only three relations last of which is derived from considering the torsion. We will show that in fact $\widehat{GR}$ is isomorphic to $\mathbb{I}\Gamma$, and discuss the relationship of $\widehat{GR}$ with the groups $\widehat{GS}$ and $\widehat{GS}_{0,0}^1$ defined by [20] and [23].

Definition 3.1. *Let $\widehat{GR}$ be the set of the invertible elements $F = (\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{\mathbb{F}}_2'$ which satisfy the relations*

(I) $\quad f(x,y)f(y,x) = 1$

(II) $\quad f(x,y)x^m f(z,x)z^m f(y,z)y^m = 1$ *where* $m = (\lambda-1)/2$ $xyz = 1$

(R) $\quad g(x_{45},x_{51})f(x_{12},x_{23})f(x_{34},x_{45}) = \sigma_2^{4\rho}(\sigma_2\sigma_3\sigma_1)^{-4\rho} f(\sigma_1^2\sigma_3^2,\sigma_2)(\sigma_1\sigma_3)^{4\rho}$

where $\rho = \rho_2(F)$ and $\rho_2 : \widehat{GT} \to \widehat{\mathbb{Z}}$ is the Kummer character associated to $\sqrt[n]{2}$, $g \in \widehat{F}_2$ is the unique element such that $f(x,y) = g(y,x)^{-1}g(x,y)$ and relation (R) *lives in* $\widehat{\Gamma}_{0,[5]}$.

Proposition 3.2. *Consider $F \in \widehat{GT}$. Then F satisfies relations* (III') *and* (IV) *from the definition of $\mathbb{I}\Gamma$ – see section 1, if and only if F satisfies the single relation* (R).

The proof is a generalization of the result of L. Schneps in [20] which establish that some *restricted* relations $(\star)$, $(\star\star)$ and (I) are equivalent to relations (II),(III') and (IV) in $\mathbb{I}\Gamma$.

Proof. Let us show that relation (R) implies relation (IV). We denote by σ_i the usual braid generators of the mapping class groups.

As relation (R) is pure with respect to the fifth string in $\Gamma_{0,[5]}$, we apply the morphism defined by erasing the fifth string. Since f belongs to $\widehat{F}_2'$ and by relation $\sigma_1^2 = \sigma_3^2$ in $\widehat{\Gamma}_{0,[4]}$ relation (R) becomes

$$\sigma_2^{4\rho_2(F)}(\sigma_2\sigma_3\sigma_1)^{-4\rho_2(F)} f(\sigma_1^4,\sigma_2)\sigma_1^{8\rho_2(F)} = f(\sigma_1^2,\sigma_2^2)$$

with $x_{12} = \sigma_1^2$ et $x_{23} = \sigma_2^2$. Relation (I) then leads to

$$f(\sigma_2,\sigma_1^4) = \sigma_1^{8\rho_2(F)} f(\sigma_2^2,\sigma_1^2)\sigma_2^{4\rho_2(F)}(\sigma_2\sigma_1\sigma_3)^{-4\rho_2(F)}.$$

By braid relations and relation $\sigma_1^2 = \sigma_3^2$ one gets $(\sigma_2\sigma_1\sigma_3)^4 = (\sigma_2\sigma_1)^6$. Hence

$$f(\sigma_2,\sigma_1^4) = \sigma_1^{8\rho_2(F)} f(\sigma_2^2,\sigma_1^2)\sigma_2^{4\rho_2(F)}(\sigma_2\sigma_1)^{-6\rho_2(F)}$$

which gives relation (IV) by symmetry on generators σ_1 and σ_2.

To establish relation (III') let us consider relation (IV) for which we just proved that it is implied by (R)

$$\text{(IV)} \qquad f(\sigma_1,\sigma_2^4)=\sigma_2^{8\rho_2(F)}f(\sigma_1^2,\sigma_2^2)\sigma_1^{4\rho_2(F)}(\sigma_1\sigma_2)^{-6\rho_2(F)}.$$

By braid relations this relation can be written in the subgroup $<\sigma_1,\sigma_2^2>$ of $\widehat{\Gamma}_{0,[4]}$

$$f(\sigma_1,\sigma_2^4)=\sigma_2^{8\rho_2(F)}f(\sigma_1^2,\sigma_2^2)\sigma_1^{4\rho_2(F)}(\sigma_1\sigma_2^2)^{-4\rho_2(F)}.$$

Let us consider the image through the homomorphism ψ defined by $\sigma_1\mapsto\sigma_2$ and $\sigma_2^2\mapsto\sigma_1\sigma_3$ in the corresponding subgroups of $\widehat{\Gamma}_{0,[4]}$ and $\widehat{\Gamma}_{0,[5]}$

$$f(\sigma_2,\sigma_1^2\sigma_3^2)=(\sigma_1\sigma_3)^{4\rho_2(F)}f(\sigma_2^2,\sigma_1\sigma_3)\sigma_2^{4\rho_2(F)}(\sigma_2\sigma_1\sigma_3)^{-4\rho_2(F)},$$

then

$$f(\sigma_1^2\sigma_3^2,\sigma_2)=(\sigma_2\sigma_1\sigma_3)^{4\rho_2(F)}\sigma_2^{-4\rho_2(F)}f(\sigma_1\sigma_3,\sigma_2^2)(\sigma_1\sigma_3)^{-4\rho_2(F)}.$$

Identifying the left hand side with a term of relation (R) we obtain the relation (III')

$$f(\sigma_1\sigma_3,\sigma_2^2)=g(x_{45},x_{51})f(x_{12},x_{23})f(x_{34},x_{45}).$$

Conversely one checks by using the homomorphism ψ above that the two relations (III') and (IV) imply the single relation (R). □

Remark 3.3. The key morphism ψ in the previous proof can be given the following quotient-type meaning in genus one. Let us consider $\gamma\in\Gamma_{1,[2]}$ the order 2 rotation whose axis goes through the hole of the torus (we refer to section 5.2 for figures and notation) and $H\subset\Gamma_{1,[2]}$ the orbifold fundamental group associated to the special locus of γ – i.e. substack of curves admitting γ as automorphism. One defines the quotient morphism $H\to\Gamma_{1,1}$, which also not onto, admits a section $H'\to H$ from its image $H'\subset\Gamma_{1,1}$ following [19]. One can then show that H', resp. H, is the subgroup generated by σ_1 and σ_2^2, resp. by σ_2 and $\sigma_1\sigma_3$; and the section is explicitly given by ψ.

By theorem 2.2 ,we then obtain the following corollary.

Corollary 3.4. *The group $\widehat{GR}$ is isomorphic to $\mathbb{I}\Gamma$, contains the absolute Galois group $G_{\mathbb{Q}}$, and thus acts on all the full mapping class groups $\Gamma_{g,[n]}$.*

Remark 3.5. Like relation (R), the defining relations of the groups $\widehat{GS}_{0,0}^1$ and $\widehat{GS}$ come from the requirement of respecting some quotient-type morphisms between moduli spaces – morphisms coming from quotienting topological surfaces by finite order diffeomorphisms – in dimension one and two in genus zero. This geometric aspect emerges more clearly from this definition than from that of $\mathbb{I}\Gamma$ even if the two groups are isomorphic.

In [23] H. Tsunogai presents two complete forms of the relations ($\star$) and ($\star\star$) for which he proves that they are satisfied by the absolute Galois group $G_{\mathbb{Q}}$. One can show that H. Tsunogai's generalized ($\star$) implies Eq. (R) by generalizing the result of [20] that ($\star$) implies Eq. (R) with $\rho=0$. This shows that $\widehat{GS}$ is a subgroup of $\widehat{GR}\simeq\mathbb{I}\Gamma$.

4. Profinite and Discrete Torsion, Serre Theory

We now turn to the study of p-torsion elements of the profinite full mapping class groups in genus 1. We will proceed by reducing the profinite p-torsion to the discrete p-torsion, using a *goodness* property of certain mapping class groups. Recall that a discrete group G is said to be *good* if it satisfies the following cohomological property

$$H^n(\widehat{G}, M) \tilde{\rightarrow} H^n(G, M) \text{ for all } n \text{ and for any } \mathbb{Z}[G]- \text{ torsion module } M. \tag{7}$$

In what follows, we make broad use of the following general result, developed in [3] following an idea of J. P. Serre [21] and based on a series of results of [6], [2, chap. X] and [12].

Proposition 4.1. *Let $G = H \rtimes \langle g\rangle$ be a discrete group of finite virtual cohomological dimension, where g is an element of prime order p acting on a torsion-free group H by conjugacy. Suppose that G is good and possesses a finite number of conjugacy classes of group of order p. Then the following non-abelian sets correspond bijectively*

$$H^1(\langle g\rangle, \widehat{H}) \simeq H^1(\langle g\rangle, H).$$

Let us briefly explain in which manner this proposition is related to the general problem of determining the profinite torsion of a profinite group $\widehat{G}$. We recall the following $(\star)$ property.

Let $\{G_i\}_I$ be a finite family composed of one representative for each conjugacy classe of the finite maximal subgroups of G such that:

(1) *Every finite subgroup of G is conjugate to a subgroup of one of G_i;*
(2) *For $i \neq j$ or $g \notin G_i$ then $G_j \cap gG_ig^{-1} = \{1\}$.* $(\star)$

In order to characterize the profinite torsion of $\widehat{G}$ for a good group G, it is sufficient to establish property $(\star)$ for a family of *discrete* groups as it implies the same property for the profinite completion $\widehat{G}$ relatively to the family $\{\widehat{G}_i\}_I$ – see [21]. for proof. As detailed in [3], our application of proposition 4.1 to the genus one mapping class groups relies on the $(\star)$ property. However, the non-intersection property is known not to hold for conjugacy classes of general finite subgroups; for this reason we restrict our attention to subgroups of prime order.

Proposition 4.2. *The mapping class groups $\Gamma_{1,n}$ and $\Gamma_{1,[n]}$ are good groups for all $n \geqslant 1$.*

The proof uses the fact that the goodness property is preserved under group extension – see for example [15].

Proof. Recall that $\Gamma_{1,1} = SL_2(\mathbb{Z})$ and that the congruence subgroup $\Gamma_2(3) \subset SL_2(\mathbb{Z})$ is free, hence good. Consider the following exact sequence

$$1 \to \Gamma_2(3) \to SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/3\mathbb{Z}) \to 1$$

where $SL_2(\mathbb{Z}/3\mathbb{Z})$ is good as a finite group. Then $\Gamma_{1,1}$ is good by [15]. By the same argument and induction on n applied to the point erasing Birman exact sequence

$$1 \to \pi_1(S_{1,n-1}) \to \Gamma_{1,n} \to \Gamma_{1,n-1} \to 1.$$

and the permutation exact sequence

$$1 \to \Gamma_{1,n} \to \Gamma_{1,[n]} \to \mathfrak{S}_n \to 1.$$

we conclude that $\Gamma_{1,n}$ then $\Gamma_{1,[n]}$ are good groups. □

We will use these two results in the following sections to determine the profinite p-torsion in genus one.

5. Genus One and Genus Zero related Cases

We begin the study of the $\mathbb{I}\Gamma$ action on conjugacy classes of torsion elements of genus one mapping class groups $\widehat{\Gamma}_{1,[n]}$. In this section we first deal with $\widehat{\Gamma}_{1,1}$ and $\widehat{\Gamma}_{1,[2]}$ which actually carry a $\widehat{GT}$ action, and then consider certain rotation in $\widehat{\Gamma}_{1,[n]}$ closely related to the genus zero situation.

We identify conjugacy classes of prime order torsion elements and deduce the explicit action of Grothendieck-Teichmüller group by the use of the cohomological proposition 4.1.

5.1. Torsion of $\widehat{\Gamma}_{1,1}$, action of $\widehat{GT}$.

The mapping class group of dimension one in genus one is the usual modular group of elliptic curves $\Gamma_{1,1} = SL_2(\mathbb{Z})$.

The following result is well-known and is recalled in the context of Dehn twists.

Proposition 5.1. *A finite order element of* $\Gamma_{1,1}$ *is of order* 2*,* 3*,* 4 *or* 6 *and conjugate to a power of*

$$\gamma_4 = \sigma_1\sigma_2\sigma_1 \qquad \gamma_6 = \sigma_1\sigma_2$$

where σ_1 *and* σ_2 *are usual generators of* B_3*. Any finite maximal subgroup of* $\Gamma_{1,1}$ *is isomorphic to* $\mathbb{Z}_4$ *or* $\mathbb{Z}_6$ *and there exists a single conjugacy class for each given order.*

Proof. As $\Gamma_{1,1} = SL_2(\mathbb{Z})$ we have the following amalgamation representation $\Gamma_{1,1} = \mathbb{Z}_4 \star_2 \mathbb{Z}_6$. Following Serre's theorem on trees [22, theorem 8], any finite subgroup is contained in $\mathbb{Z}_4$ or $\mathbb{Z}_6$.

The braid group quotient representation of $\Gamma_{1,1}$ through $SL_2(\mathbb{Z})$ is well-known

$$SL_2(\mathbb{Z}) \simeq \frac{B_3}{\langle(\sigma_1\sigma_2)^6 = 1\rangle}.$$

The result on conjugacy classes is straightforward since $SL_2(\mathbb{Z})$ is a free group. □

Using braid relations, one notices that the following elements

$$\gamma_2 = (\sigma_1\sigma_2)^3 \qquad \gamma_3 = (\sigma_1\sigma_2)^2$$

are respectively of order 2 and 3, and satisfy the following relations

$$\gamma_2 = \gamma_4^2 \qquad \gamma_6 = \gamma_4^2\gamma_3^{-1}.$$

Similarly to the genus zero case – see [3], property 4.1 does not hold for *arbitrary finite subgroups* in the mapping class group $\Gamma_{1,1}$ since a counter-example is given by:

$$\langle\gamma_4\rangle \cap g\langle\gamma_6\rangle g^{-1} \supset \langle\gamma_2\rangle$$

which contradicts the $(\star)$ property – see remark below proposition 4.1. Equivalently this comes from the fact that every elliptic curve admits an elliptic automorphism.

Because of this problem of non-trivial intersection of finite subgroups, we only deal with prime order torsion in this paper.

Proposition 5.2. *Let γ be a torsion element of order 2 or 3 of $\widehat{\Gamma}_{1,1}$. Then γ is conjugate to one of the following*

$$\gamma_2 = (\sigma_1\sigma_2)^3 \qquad \gamma_3 = (\sigma_1\sigma_2)^2$$

where σ_i is a generator of $\widehat{B}_3$.

Proof. In order to apply proposition 4.1 to establish that profinite torsion of prime order of $\widehat{\Gamma}_{1,1}$ is conjugate to the one of the discrete $\Gamma_{1,1}$, it is sufficient to give a normal finite index subgroup $H \subset \Gamma_{1,1}$ which is torsion free and stable under conjugation by γ_2 (resp. γ_3).

One takes H to be the free normal congruence subgroup $\Gamma_2(3)$. Then for the element $\gamma = \gamma_2$ or γ_3, the group $G = H \rtimes \langle\gamma\rangle$ is good and possesses only a finite number of conjugacy classes of order 2 or 3 by proposition 5.1. □

We can now establish the form of the $\widehat{GT}$-action on these elements.

Proposition 5.3. *The Grothendieck-Teichmüller group $\widehat{GT}$ acts by λ-conjugacy on prime order torsion of $\widehat{\Gamma}_{1,1}$.*

Proof. Let $\gamma \in \widehat{\Gamma}_{1,1}$ be a torsion element of prime order and $F \in \widehat{GT}$. As γ and $F(\gamma)$ have same order, then following the previous proposition there exists $g \in \widehat{\Gamma}_{1,1}$ and $k \in \widehat{\mathbb{Z}}$ such that

$$F(\gamma) = g\gamma^k g^{-1}.$$

From the braid quotient representation of $\Gamma_{1,1}$, one identifies the abelianization $\Gamma_{1,1}^{ab}$ with the cyclic group $\mathbb{Z}_{12}$. It is generated by σ_1^{ab} image of the twist σ_1 – or by any twist along a non-separating curve, since two such twists are conjugate. As $\sigma_1 = \gamma_3\gamma_4^{-1}$, one obtains

$$\Gamma_{1,1}^{ab} = \langle\gamma_3^{ab}\rangle \times \langle\gamma_4^{ab}\rangle.$$

which implies that any torsion element keeps the same order in the abelianization. Moreover

$$\gamma_2^{ab} = (\sigma_1^{ab})^6 \qquad \gamma_3^{ab} = (\sigma_1^{ab})^4.$$

Since the $\widehat{GT}$ action commutes with abelianization, the relation $F(\gamma) = g\gamma^k g^{-1}$ gives

$$F(\gamma)^{ab} = (\gamma^{ab})^k \tag{8}$$

We now compute the power k in this expression for any torsion element of prime order.

Let us first consider $\gamma_3 = \sigma_1\sigma_2\sigma_1\sigma_2$. One obtains $\gamma_3^{ab} = \sigma_1^4$, and since $\widehat{GT}$ acts by λ-conjugacy on twists along simple closed curves – see theorem 2.2, then $F(\gamma_3^{ab}) = \sigma_1^{4\lambda}$. Relation (8) in $\Gamma_{1,1}^{ab}$ gives $\sigma_1^{4k} = \sigma_1^{4\lambda}$ from which we deduce

$$k = \lambda \mod 3.$$

For $\gamma_2 = (\sigma_1\sigma_2)^3$ a similar computation leads to the equation

$$k = \lambda \mod 2.$$

□

Remark 5.4. Consider the reduced mapping class group $\bar{\Gamma}_{0,[4]}$ defined as the fundamental group of $\mathbb{P}^1 - \{0,1,\infty\}/\mathfrak{S}_3$. H. Tsunogai and H. Nakamura obtain a full explicit expression of the $G_{\mathbb{Q}}$-action on *discrete* torsion elements of $\bar{\Gamma}_{0,[4]}$ by precising the conjugating elements – see [23], [18] and [16]. Let $\bar{\Gamma}_{1,1}$ denote the quotient of $\Gamma_{1,1}$ by the elliptic involution. Since $\bar{\Gamma}_{1,1}$ is isomorphic to $\bar{\Gamma}_{0,[4]}$, one deduces from their work an explicit Galois action on discrete torsion elements of $\bar{\Gamma}_{1,1}$.

5.2. **Torsion of $\widehat{\Gamma}_{1,[2]}$, action of $\widehat{GT}$.** Let us recall the following isomorphism between the mapping class group $\Gamma_{1,[2]}$ and a central extension of a braid quotient.

Proposition 5.5. *The full mapping class group $\Gamma_{1,[2]}$ admits the presentation*

$$\Gamma_{1,[2]} \simeq B_4/Z \times \mathbb{Z}/2\mathbb{Z}$$

where Z denotes the center of the braid group B_4.

Proof. This identification comes from the usual exact sequence

$$1 \to \Gamma_{1,2} \to \Gamma_{1,[2]} \to \mathfrak{S}_2 \to 1$$

where one identifies each term.

Let s_i be the twists whose supports are such as on the Fig. 3, with the morphism

$$B_4 \to \Gamma_{1,2}$$
$$\sigma_i \to s_i$$

Such twists satisfy the usual braid relations together with the following one (see F. Luo [14])

$$(s_1s_2s_3)^4 = 1$$

which is the center relation in B_4. This gives the isomorphism between $\Gamma_{1,2}$ and the braid quotient.

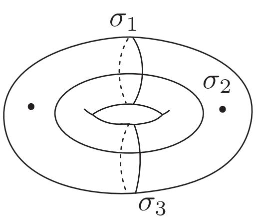


FIGURE 3. Generators of $\Gamma_{1,2}$.

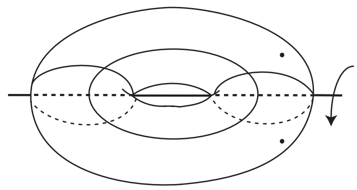


FIGURE 4. Elliptic involution τ.

Moreover, an element τ in $\Gamma_{1,[2]}$ whose image generates $\mathfrak{S}_2$ is given by the elliptic involution of Fig. 4. As a pure element fixes every point, one has $\Gamma_{1,2} \cap \langle\tau\rangle = \{1\}$.

Eventually one checks on the other generators of $\Gamma_{1,[2]}$ and on a homology basis of $S_{1,2}$ that τ commutes with every s_i. □

From this representation, one deduces the following list of torsion elements.

Proposition 5.6. *Let γ be a torsion element of $\Gamma_{1,[2]}$. Then γ is conjugate to one of the following*

$$\begin{array}{lll} \gamma_{2,p} = (\sigma_1\sigma_2\sigma_3)^2 & \gamma_{3,p} = \sigma_1\sigma_2\sigma_3\sigma_1 & \gamma_{4,p} = (\sigma_1\sigma_2\sigma_3) \\ \gamma_2 = \gamma_{2,p}\tau & \gamma_4 = \gamma_{4,p}\tau & \gamma_6 = \gamma_{3,p}\tau \end{array}$$

where γ_k, resp. $\gamma_{k,p}$ denotes an element, resp. a pure element, of order k.

Let us now establish that profinite torsion of prime order is related to the discrete one.

Proposition 5.7. *Let $\gamma \in \widehat{\Gamma}_{1,[2]}$ be a torsion element of prime order. Then γ is conjugate to one of the finite discrete order elements of the previous proposition.*

It is sufficient to provide a torsion free normal subgroup H of finite index in $\Gamma_{1,[2]}$. Remark that unlike the case of genus zero, the pure mapping class group $\Gamma_{1,2}$ contains torsion elements.

Proof. Let H be the intersection of the conjugates of $\Gamma_{0,5}$ inside $\Gamma_{1,2}$. Indeed we have the inclusions

$$\Gamma_{0,5} \hookrightarrow \Gamma_{1,2} \hookrightarrow \Gamma_{0,[5]},$$

which shows that $\Gamma_{0,5}$ is of a finite index in $\Gamma_{1,2}$; therefore so is H. Furthermore H is normal and torsion free by definition.

Let γ be a prime order p torsion element, then $G = H \rtimes \langle\gamma\rangle$ admits a finite number of conjugacy classes of order p as subgroup of $\Gamma_{1,[2]}$. This follows from the finite number of conjugacy classes in $\Gamma_{1,[2]}$ established in general in theorem 6.1. The proposition then results from the proposition 4.1. □

Using analogous methods as in the previous section we obtain the action on pure elements, whereas full elements of $\Gamma_{1,[2]}$ are considered separately.

Proposition 5.8. *The Grothendieck-Teichmüller group $\widehat{GT}$ acts by λ-conjugacy on the torsion of order 2 and 3 of $\widehat{\Gamma}_{1,[2]}$.*

Proof. Following proposition 5.7 and the exact argument of proposition 5.3 in $\Gamma_{1,1}$, the action of an element $F = (\lambda, f) \in \widehat{GT}$ on a prime order torsion element $\gamma \in \widehat{\Gamma}_{1,[2]}$ is given by

$$F(\gamma) = g\gamma^k g^{-1}$$

where $g \in \widehat{\Gamma}_{1,[2]}$ and $k \in \widehat{\mathbb{Z}}^*$ is a power to identify.

First suppose that $\gamma \in \widehat{\Gamma}_{1,2}$ is pure. Since $\Gamma_{1,2} \simeq B_4/Z$, one obtains $\Gamma_{1,2}^{ab} \simeq \mathbb{Z}/12\mathbb{Z}$ and pure torsion element keep same order in the abelianization group. Using the expression of the $\widehat{GT}$ action on braid generators, we identify $k = \lambda$.

Then suppose $\gamma \in \Gamma_{1,[2]} - \Gamma_{1,2}$, which is then of order 2 following proposition 5.6. Since the action of $\widehat{GT}$ does not change the permutation associated to an element, then $k = 1 \mod 2$. As $\lambda \in \widehat{\mathbb{Z}}^*$, it is invertible modulo 2 which concludes the proof. □

In this section, a presentation of $\Gamma_{1,[2]}$ related to the genus zero case has been used – through a quotient of braid group and the elliptic involution – in order to give the action on prime order torsion elements of $\Gamma_{1,[2]}$. In the following section, we introduce another specific type of torsion element of $\Gamma_{1,[n]}$, which comes also in a certain way from genus zero. From this, we will deduce the $\widehat{GR}$ action on this type of torsion element, and section 6 will complete the general case.

5.3. **Genus zero rotation into $\widehat{\Gamma}_{1,[n]}$, action of $\widehat{GR}$.** We give the $\widehat{GR}$ action on particular torsion elements of $\widehat{\Gamma}_{1,[n]}$ which permute every points – marked or not – of the surface. These elements correspond to translations or equivalently to geometric rotations with axis going through the hole of the torus.

We first establish that these torsion elements come from a genus zero configuration, then prove the expected $\widehat{GR}$ action implied by this relation. Let us note that this section makes use of proposition 4.1 through the genus zero case, as it is used in [3] to establish the $\widehat{GT}$-action on prime order torsion elements of $\widehat{\Gamma}_{0,[n]}$.

Recall that $\Gamma_{g,n}^b$ denotes the pure mapping class group associated to a surface of genus g with n marked point and b boundaries.

Proposition 5.9. *Let $\varepsilon_n \in \widehat{\Gamma}_{0,[n]}^2$ be the torsion element of order n of Fig. 5. Consider the morphism – with notation of Fig. 5:*

$$\begin{aligned} \psi : \widehat{\Gamma}_{0,[n]}^2 &\to \widehat{\Gamma}_{1,[n]} \\ b_i &\mapsto b \qquad 1 \leqslant i \leqslant 2 \\ \sigma_i &\mapsto \tau_i \qquad 1 \leqslant i \leqslant n \end{aligned}$$

Then $r_n := \psi(\varepsilon_n)$ is an order n element of $\widehat{\Gamma}_{1,[n]}$.

Proof. The mapping ψ is a well-defined group morphism given on generators of the mapping class groups.

In fact ε_n is the rotation in Fig. 6, so $\varepsilon_n \in \widehat{\Gamma}_{0,[n]}^2$, and the homomorphism $\psi : \widehat{\Gamma}_{0,[n]}^2 \to \widehat{\Gamma}_{1,[n]}$ has kernel $Ker(\psi) = \widehat{\mathbb{Z}}$ that is generated by $b_1 b_2^{-1}$. Hence $\varepsilon_n^k \notin Ker(\psi)$ for all $k < n$ and r_n has order n as an element of $\widehat{\Gamma}_{1,[n]}$. □

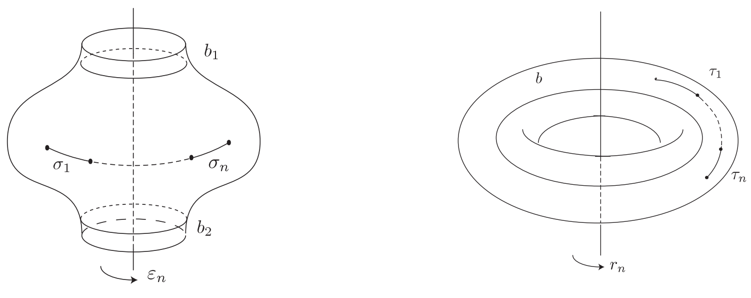


FIGURE 5. Rotation in genus zero and one: identify b_1 and b_2.

The expression of the action of $\widehat{GR}$ on this element r_n in $\widehat{\Gamma}_{1,[n]}$ can be deduced from the $\widehat{GT}$ action on torsion in genus zero – see [3].

Proposition 5.10. *Let p be a prime dividing n and $a = n/p$. Consider $F \in \widehat{GR}$. Then r_n^a is of prime order p and F acts on it by λ-conjugacy.*

$$F(r_n^a) = g\, r_n^{a\lambda}\, g^{-1}$$

where $g \in \widehat{\Gamma}_{1,[n]}$.

Proof. Let $F \in \widehat{GR}$. Following the previous proposition, the element $r_n^a \in \widehat{\Gamma}_{1,[n]}$ comes from an element $\varepsilon_n^a \in \widehat{\Gamma}^2_{0,[n]}$ of order n through the morphism ψ.

Following corollary 2.5 and corollary 3.4 the following diagram commutes up to an inner automorphism

$$\begin{array}{ccc} \widehat{\Gamma}^2_{0,[n]} & \xrightarrow{\psi} & \widehat{\Gamma}_{1,[n]} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \widehat{\Gamma}^2_{0,[n]} & \xrightarrow{\psi} & \widehat{\Gamma}_{1,[n]} \end{array}$$

where action on first, resp. second, column is the usual $\widehat{GT}$ action defined in genus zero, resp. in genus one in theorem 2.2.

Following the $\widehat{GT}$ action on prime order torsion elements in genus zero established in [3] (that uses proposition 4.1 of this paper), we obtain

$$F(\varepsilon_n^a) = g\varepsilon_n^{a\lambda} g^{-1},$$

and following the previous commutative diagram

$$F(r_n^a) = h\psi(g)\, r_n^{a\lambda}\, \psi(g)^{-1}h^{-1}.$$

□

Remark 5.11. Computation of this action by abelianization like in the case $(g,n) = (1,2)$ fails as $(r_n)^{ab} = 1$.

The remaining prime order torsion elements of $\widehat{\Gamma}_{1,[n]}$ do not come from genus zero. In the following section, we describe their conjugacy classes and we establish the $\widehat{GR}$ action by λ-conjugacy for these more complicated elements.

6. Action of $\widehat{GR}$ on Prime Torsion of $\pi_1^{geom}(\mathcal{M}_{1,[n]})$

We now turn to the $\widehat{GR}$-action on prime order torsion of $\widehat{\Gamma}_{1,[n]}$. We first identify conjugacy classes of these elements with conjugacy classes of discrete ones in $\Gamma_{1,[n]}$. So far we have determined the $\widehat{GR}$-action on all prime order torsion elements of $\widehat{\Gamma}_{1,1}$ and $\widehat{\Gamma}_{1,2}$, and also for the genus zero rotation-type elements of $\widehat{\Gamma}_{1,[n]}$ for all n. In this section we deal with all other prime torsion elements of $\widehat{\Gamma}_{1,[n]}$.

6.1. **Prime order torsion of $\widehat{\Gamma}_{1,[n]}$.** Following the Nielsen realization result [9], any torsion element of $\Gamma_{1,[n]}$ corresponds to an automorphism of an elliptic curve. As the topological torus admits three different types of analytic structure, the automorphism group of a genus one surface X is given by

$$Aut(X) = T \rtimes G_0$$

where T is the translation group by a point of X and G_0 is isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$ according to the j-invariant of X.

In the following proposition we completely classify the different torsion conjugacy classes of $\Gamma_{1,[n]}$.

Proposition 6.1. *A torsion element of the mapping class group $\Gamma_{1,[n]}$ belongs to one of the* 21 *conjugacy classes given in table 1.*

The approach we take is to use the Riemann-Hurwitz formula and to count fibers over elliptic points of the quotient surface, to obtain necessary conditions on the signature. As finite automorphisms are induced by diffeomorphisms of the surface realized by a specific distribution of the points, we then eliminate cases that contradict this property. We need the following lemma.

Lemma 6.2. *Let $\Gamma^P_{1,[n]}$ denote the subgroup of diffeomorphism classes of a topological surface S which fix a given point $P \in S$, and let $\Gamma^P_{1,[n]} \to \Gamma_{1,1}$ be the homomorphism given by erasing the n marked points different from P, and marking P if it is not already marked. Then the kernel is torsion free.*

Proof. The kernel of this morphism is well known to be the genus one braid group $B_{1,1}(n-1)$ (resp. $B_{1,1}(n)$) when the fixed point P is marked (resp. is not). By definition, it is the topological fundamental group of space of the configuration space $(T^n - \Delta)/\mathfrak{S}_n$, where T is the once punctured torus, and Δ is the fat diagonal. Now, the sequence

$$1 \to P_{1,1}(n) \to B_{1,1}(n) \to \mathfrak{S}_n \to 1$$

is exact, where $P_{1,1}(n) = \pi_1(T^n - \Delta)$ is torsion free. But $\mathfrak{S}_n$ acts without fixed points on $T^n - \Delta$, so the quotient is an algebraic variety rather than a stack, i.e. its fundamental group $B_{1,1}(n)$ acts freely on the universal cover. But the universal cover is contractible because this space is a $K(\pi, 1)$. A finite order element acting on a contractible space must have a fixed point by Brouwer's theorem, which shows that $B_{1,1}(n)$ and $B_{1,1}(n-1)$ are torsion free. □

Proof of proposition 6.1. Let us consider $S_{1,n}$ a topological torus with n marked points, γ a torsion element of finite order and let us denote $S'_{g,m} = S_{1,n}/\langle\gamma\rangle$ the quotient surface of genus $g \le 1$ with m marked points. Let us recall the Riemann-Hurwitz formula for genus one.

$$n = k(2g - 2 + m) + k \sum_{ell} (1 - 1/e_x) \tag{9}$$

where k is the order of $G = \langle\gamma\rangle$ and e_x are elliptic ramification indices of a point $x \in S'_{g,m}$. We denote by r_ℓ the number of elliptic points of order ℓ.

Table 1. Torsion of $\Gamma_{1,[n]}$.

	Branching (t_1,t_2,t_3,t_4,t_6)	Signature (Σ_γ)
Elliptic involution	$(4,(n-4)/2,0,0,0)$	$\left(0,\frac{n-4}{2}+4,\varnothing\right)$
	$(3,(n-3)/2,0,0,0)$	$\left(0,\frac{n-3}{2}+3,\{2\}\right)$
	$(2,(n-2)/2,0,0,0)$	$\left(0,\frac{n-2}{2}+2,\{2,2\}\right)$
	$(1,(n-1)/2,0,0,0)$	$\left(0,\frac{n-1}{2}+1,\{2,2,2\}\right)$
	$(0,n/2,0,0,0)$	$\left(0,\frac{n}{2},\{2,2,2,2\}\right)$
order 3	$(3,0,n/3,0,0)$	$\left(0,\frac{n-2}{3}+3,\varnothing\right)$
	$(2,0,(n-3)/3,0,0)$	$\left(0,\frac{n-3}{3}+2,\{3\}\right)$
	$(1,0,(n-1)/3,0,0)$	$\left(0,\frac{n-1}{3}+1,\{3,3\}\right)$
	$(0,0,n/3,0,0)$	$\left(0,\frac{n}{3},\{3,3,3\}\right)$
order 4	$(2,0,0,(n-2)/4,0)$	$\left(0,\frac{n-2}{4}+2,\{2\}\right)$
	$(1,1,0,(n-3)/4,0)$	$\left(0,\frac{n-3}{4}+2,\{4\}\right)$
	$(1,0,0,(n-1)/4,0)$	$\left(0,\frac{n-1}{4}+1,\{2,4\}\right)$
	$(0,1,0,(n-2)/4,0)$	$\left(0,\frac{n-2}{4}+1,\{4,4\}\right)$
	$(0,0,0,n/4,0)$	$\left(0,\frac{n}{4},\{2,4,4\}\right)$
order 6	$\left(0,0,0,0,n/6\right)$	$\left(0,\frac{n}{6},\{2,3,6\}\right)$
	$\left(0,1,1,0,\left(n-5\right)/6\right)$	$\left(0,\frac{n-5}{6}+2,\{6\}\right)$
	$\left(1,0,0,0,\left(n-1\right)/6\right)$	$\left(0,\frac{n-1}{6}+1,\{2,3\}\right)$
	$\left(1,0,1,0,\left(n-4\right)/6\right)$	$\left(0,\frac{n-4}{6}+2,\{2\}\right)$
	$\left(1,1,0,0,\left(n-3\right)/6\right)$	$\left(0,\frac{n-3}{6}+2,\{3\}\right)$
	$\left(1,1,1,0,\left(n-6\right)/6\right)$	$\left(0,\frac{n-6}{6}+3,\varnothing\right)$
Rotation type elements	$\varnothing$	$\left(1,\frac{n}{k},\varnothing\right)$

Let us first suppose that γ fixes no point on the surface. Then the Riemann-Hurwitz formula implies $g = 1$, and since fixed points corresponds to elliptic ramification points, the previous formula (9) gives

$$n = km.$$

One deduces that $ord(\gamma)$ divides n and γ corresponds to a rotation as automorphism of Riemann surface – such rotation type elements are studied in the previous section 5.3.

Let us now suppose that γ admits one fixed point, i.e. induces an automorphism of an elliptic curve. In this case $g = 0$ and $k \in \{2,3,4,6\}$. Let us denote by t_q the number of fixed points of $S'_{g,m}$ for which the inertia group is of order q dividing k. By counting points in fibers one obtains

$$n = t_1 + \sum_{q|ord(G)} t_q q \tag{10}$$

$$m = t_1 + \sum_{q|ord(G)} t_q. \tag{11}$$

We reduce the formulae (10) and (11) in (9) to

$$(k-1)t_1 + (k-2)t_2 + (k-3)t_3 + (k-4)t_4 + (k-6)t_6 + \frac{k}{2}r_2 + 2\frac{k}{3}r_3 + 3\frac{k}{4}r_4 + 5\frac{k}{6}r_6 = 2k,$$

which gives 38 admissible signatures. By lemma 6.2, on the surface with forgotten marked points, γ induces an elliptic automorphism whose branching data belongs to

$$\{\{2,2,2,2\},\{2,4,4\},\{3,3,3\},\{2,3,6\}\}.$$

Hence one excludes signatures Σ_γ amongst the 38 previous ones which do not correspond to these branching data, and we obtain table 1.

More precisely, for a given inertia data $(t_1, t_2, t_3, t_4, t_6)$, one fixes an analytic structure of the topological torus and then one marks some points amongst elliptic or ordinary points. Figs. 7 and 6 below give two examples realizing an order 4 and an order 6 torsion element. For order 4, the non-marked vertices (resp. middle of sides) give an order 4 (resp. order 2) elliptic point, for order 6, the non-marked middle of sides give an order 2 elliptic point.

The fact that only one conjugacy class is assigned to each signature comes from the fact that there exists only a single topological way to distribute the marked points on the torus to realize the automorphism. □

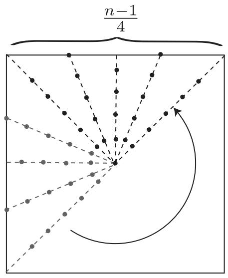


FIGURE 6. Signature $\{0, \frac{n-1}{4}+1, \{2,4\}\}$.

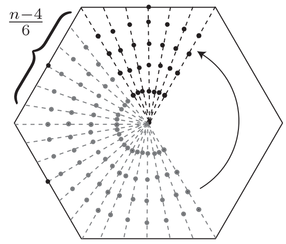


FIGURE 7. Signature $\{0, \frac{n-4}{6}+2, \{2\}\}$.

We now establish the expected result on profinite torsion of $\widehat{\Gamma}_{1,[n]}$.

Theorem 6.3. *Every torsion element of prime order of the profinite groups* $\widehat{\Gamma}_{1,[n]}$ *is conjugate to an element of the discrete group* $\Gamma_{1,[n]}$.

Proof. Following property 4.1, it is sufficient to provide a finite index normal subgroup H of $\Gamma_{1,[n]}$ which is torsion free and such that for $g \in \Gamma_{1,[n]}$ of order p prime, the subgroup $G = H \rtimes \langle g \rangle \subset \Gamma_{1,[n]}$ is good and possesses a finite number of conjugacy classes of order p.

First remark that one can suppose that H is normal in $\Gamma_{1,[n]}$ by substituting its normal core $\cap_g gHg^{-1}$ which has finite index if H has finite index. The construction of a torsion free subgroup H of $\Gamma_{1,[n]}$ is then given by induction on n and by the use of the point erasing exact sequence

$$1 \longrightarrow \pi_1(S_{1,n-1}) \longrightarrow \Gamma_{1,n} \xrightarrow{\psi} \Gamma_{1,n-1} \longrightarrow 1$$

since $\pi_1(S_{1,n-1})$ is torsion free. Let us suppose that $\Gamma_{1,n-1}$ possesses a finite index torsion free subgroup G, and prove that $H = \psi^{-1}(G) \subset \Gamma_{1,n}$ is torsion free. If there exists a torsion element $\gamma \in H$ of order k, then $\psi(x)^k = 1$ and $\psi(x)$ is a torsion element in $G \subset \Gamma_{1,n-1}$ which is known to be torsion free. Hence $\psi(x) = 1$ i.e x is in the kernel $\pi_1(S_{1,n-1})$. Since this group is torsion free, it implies $x = 1$. Let us note that H has finite index in $\Gamma_{1,n}$, hence in $\Gamma_{1,[n]}$.

The induction starts with the group $\Gamma_{1,1}$ and the free congruence subgroup $\Gamma_2(3)$ of finite index.

Moreover H is a good group as the finite index subgroup of a good group, which implies that $G = H \rtimes \langle g \rangle$ is good for $g \in \Gamma_{1,[n]}$ a torsion element.

The finite number of conjugacy classes of $H \rtimes \langle g \rangle$ comes from the same property for $\Gamma_{1,[n]}$ – see table 1. □

Let us notice that this proof is general enough to establish the well-known property that every mapping class group $\Gamma_{g,[n]}$ has finite virtual cohomological dimension.

Remark 6.4. Using the Riemann-Hurwitz formula, one establishes that pure mapping class groups $\Gamma_{g,n}$ are torsion free for $n > 2g + 2$ – i.e. for $n \geqslant 5$ in our genus one case, providing a canonical torsion-free subgroup of $\Gamma_{1,[n]}$ for $n \geqslant 5$.

6.2. **Action of $\widehat{GR}$ on prime order torsion.** Using theorem 6.1 on profinite torsion and proposition 5.3 on $\widehat{GR}$-action in the case of genus one, we now establish the $\widehat{GR}$-action result on all prime order torsion of $\widehat{\Gamma}_{1,[n]}$.

Theorem 6.5. *Let $\gamma \in \widehat{\Gamma}_{1,[n]}$ be an order prime torsion element and $F \in \widehat{GR}$. Then $\widehat{GR}$ acts by λ-conjugacy*

$$F(\gamma) = g\gamma^{\lambda}g^{-1}$$

where $g \in \widehat{\Gamma}_{1,[n]}$.

Proof. Following the classification of table 1, two cases can be distinguished: cases of an elliptic element and cases of a rotation.

First suppose that γ is a power of the rotation. Then the action is given by proposition 5.10. We can then suppose that $\gamma \in \widehat{\Gamma}^{ell}_{1,[n]} \subset \widehat{\Gamma}_{1,[n]}$ i.e. that γ fixes at least one point P on the torus.

Following proposition 6.1, both γ and $F(\gamma)$ are conjugate to a power of an element of table 1. Since they have the same order and F preserves permutations following theorem 2.2, they must correspond to powers of the same element because these properties characterize the entries of table 1. We deduce that $F(\gamma)$ is conjugate to a power of γ:

$$F(\gamma) = g\gamma^{k}g^{-1}$$

where $k \in \widehat{\mathbb{Z}}^*$ and $g \in \widehat{\Gamma}_{1,[n]}$.

We now identify the power k by distinguishing two cases according to whether the fixed point P is marked or not. In the first case, let us note $\widehat{\Gamma}^P_{1,[n]}$ the subgroup of $\widehat{\Gamma}_{1,[n]}$ of elements that leave P fixed, and consider the following map

$$\widehat{\Gamma}^P_{1,[n]} \to \widehat{\Gamma}_{1,1}$$

induced by erasing all the marked points except P. Then F preserves $\widehat{\Gamma}^P_{1,[n]}$ as it preserves permutations, and the diagram below commutes:

$$\begin{array}{ccc} \widehat{\Gamma}^P_{1,[n]} & \xrightarrow{i} & \widehat{\Gamma}_{1,1} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \widehat{\Gamma}^P_{1,[n]} & \xrightarrow{i} & \widehat{\Gamma}_{1,1}. \end{array}$$

The second case then boils down to the same situation by marking P and by considering γ as an element of $\widehat{\Gamma}^P_{1,[n+1]}$. By the same permutation argument above, we then obtain an homomorphism $\widehat{\Gamma}^P_{1,[n+1]} \to \widehat{\Gamma}_{1,1}$ with the same $\widehat{GR}$-action property.

Let us denote by $\tilde{\gamma}$ the image of γ in $\widehat{\Gamma}_{1,1}$ by one of the previous morphism. Then $\tilde{\gamma}$ has same order as γ since it is induced by the same elliptic automorphism of the torus – see 6.2. This means that γ is of order 2 or 3. Following proposition 5.2 and the previous diagram commutation, the $\widehat{GR}$-action on the prime order torsion of $\widehat{\Gamma}_{1,1}$

of proposition 5.3 gives:

$$F(\tilde{\gamma}) = h\tilde{\gamma}^{\lambda}h^{-1}$$
$$h\tilde{\gamma}^{\lambda}h^{-1} = i(g)\tilde{\gamma}^{k}i(g)^{-1}.$$

One then obtains $k = \lambda \mod 2$ or 3 passing to the abelianization group $\widehat{\Gamma}^{ab}_{1,1}$ as in the proof of proposition 5.3. □

We conclude this paper by a Galois action result. Since $\widehat{GR}$ contains the absolute Galois group

$$G_{\mathbb{Q}} \hookrightarrow \widehat{GR}$$
$$\sigma \to (\chi_{\sigma}, f_{\sigma})$$

where χ_σ is the cyclotomic character. The following Galois action result on geometric torsion element – i.e. discrete torsion element of $\pi_1^{geom}(\mathcal{M}_{1,[n]})$ – is now immediate by theorem 6.5.

Corollary 6.6. *Let $\gamma \in \pi_1^{geom}(\mathcal{M}_{1,[n]})$ a* geometric *torsion element of prime order. Then the absolute Galois action on γ is given by χ_σ-conjugacy.*

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