ACTION OF THE GROTHENDIECK-TEICHMÜLLER GROUP ON TORSION ELEMENTS OF FULL TEICHMÜLLER MODULAR GROUPS IN GENUS ZERO

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Cet article présente l'action du groupe de Grothendieck-Teichmüller $\widehat{GT}$ sur les éléments de torsion d'ordre premier du groupe fondamental profini $\pi_1^{geom}(\mathcal{M}_{0,[n]})$. Nous établissons par ailleurs que les classes de conjugaison d'éléments de torsion d'ordre premier de $\widehat{\pi}_1(\mathcal{M}_{0,[n]})$ correspondent aux classes de conjugaison discrètes *de $\pi_1(\mathcal{M}_{0,[n]})$.*

In this paper we establish the action of the Grothendieck-Teichmüller group $\widehat{GT}$ on the prime order torsion elements of the profinite fundamental group $\pi_1^{geom}(\mathcal{M}_{0,[n]})$. As an intermediate result, we prove that the conjugacy classes of prime order torsion of $\widehat{\pi}_1(\mathcal{M}_{0,[n]})$ are exactly the discrete *prime order ones of the $\pi_1(\mathcal{M}_{0,[n]})$.*

1. Introduction

In this paper we establish an essential property of the action of the Grothendieck-Teichmüller group $\widehat{GT}$ on the prime order torsion elements of the profinite fundamental group $\pi_1^{geom}(\mathcal{M}_{0,[n]})$. This property shows that the $\widehat{GT}$ action on these torsion elements is of Galois type, as the absolute Galois action on geometric (discrete) inertia of $\pi_1^{geom}(\mathcal{M}_{0,[n]})$ is given by an analogous expression.

The main difficulty in our study lies in characterising the profinite torsion conjugacy classes, which is an extremely difficult problem for profinite completions in general. In the groups $\pi_1^{geom}(\mathcal{M}_{0,[n]})$, we solve this problem for prime order torsion by showing that conjugacy classes are the same as the discrete ones.

In what follows, we identify $\pi_1^{geom}(\mathcal{M}_{0,[n]})$ with the full mapping class group $\widehat{\Gamma}_{0,[n]}$, the group of oriented diffeomorphisms of the genus zero surface with n punctures modulo those isotopic to the identity.

Inertia at infinity, stack inertia and Galois action. The idea of considering torsion elements in $\widehat{\Gamma}_{0,[n]}$ comes from the general idea of A. Grothendieck's *Esquisse d'un programme* [Gro97] to study the absolute Galois group $G_{\mathbb{Q}}$ through geometric representations in algebraic fundamental groups.

More precisely, he suggests considering representations related to moduli spaces of curves with marked points $\mathcal{M}_{0,[n]}$

$$G_{\mathbb{Q}} \hookrightarrow Out(\pi_1^{geom}(\mathcal{M}_{0,[n]})),$$

– which exist following [Oda97] because the moduli spaces $\mathcal{M}_{0,[n]}$ are algebraic stacks of type Deligne-Mumford defined over $\mathbb{Q}$ – and in particular the role of the automorphism groups of such curves.

In the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,[n]}$ of $\mathcal{M}_{g,[n]}$, the divisor at infinity D_∞ decomposes into its irreducible components:

$$D_\infty = \overline{\mathcal{M}}_{g,[n]} \backslash \mathcal{M}_{g,[n]} = \cup D,$$

where the D are normal crossing divisors of codimension one in $\overline{\mathcal{M}}_{g,[n]}$. When $g = 0$, each of the associated *inertia subgroups* I_D of $\pi_1(\mathcal{M}_{0,[n]})$ is conjugate to one of the cyclic groups $<\sigma_1\sigma_2 \cdots \sigma_k>$ for $1 \leqslant k \leqslant n-3$, where the σ_i are braid generators – see below.

For pure moduli spaces $\mathcal{M}_{g,n}$, H. Nakamura proved by using generalized Grothendieck-Murre theory that *the action of the absolute Galois group $G_\mathbb{Q}$ on inertia at infinity of $\mathcal{M}_{g,n}$ is given by $\chi(\sigma)$-conjugacy action*, i.e. conjugating an inertia generator and raising it to the power $\chi(\sigma)$ – cf. [Nak97] and [Nak99].

Because of the residual finiteness of the mapping class group, torsion elements of $\Gamma_{g,[n]}$ give torsion elements of the same order in the profinite geometric fundamental group. These *geometric torsion elements* of the profinite group correspond to *stack* inertia in the following sense.

As algebraic stacks, moduli spaces $\mathcal{M}_{g,[n]}$ admit *local stack inertia groups* $I_x = Aut(x)$ composed of the finite automorphisms of the isomorphism class of an object $x \in \mathcal{M}_{g,[n]}$. Following B. Noohi [Noo04] these *geometric* automorphisms groups inject into the fundamental group

$$\omega_x : I_x \to \pi_1^{geom}(\mathcal{M}_{g,[n]}).$$

Identifying $\pi_1^{geom}(\mathcal{M}_{g,[n]})$ with the mapping class group $\widehat{\Gamma}_{g,[n]}$, such inertia groups in fact correspond exactly to finite subgroups of $\Gamma_{g,[n]}$ by a result of S. Kerckhoff [Ker83].

It is natural to ask whether *all torsion elements of $\pi_1^{geom}(\mathcal{M}_{g,[n]})$ are conjugate to the geometric ones* and whether *$G_\mathbb{Q}$ acts on torsion elements by $\chi(\sigma)$-conjugacy.* The first main result in this paper answers the first question in the case of genus zero and prime order torsion.

Theorem A. *All prime order torsion elements of $\widehat{\Gamma}_{0,[n]}$ are conjugate to geometric torsion elements.*

The approach taken is cohomological and based on an idea of J. P. Serre [Ser97] : conjugacy classes of finite subgroups of profinite completions of *good* groups can be determined by the discrete group (proposition 3.4). The main obstacle in applying Serre's theory to the non-prime power torsion elements in $\Gamma_{0,[n]}$ is the fact that distinct conjugacy classes of same order can intersect – see remark 3.4. Characterising order of torsion by cohomology properties is also a well-known difficult problem – see remark 2.9.

For the second question, one can show that the Galois action is as expected on all *geometric* torsion elements of $\widehat{\Gamma}_{0,[n]}$ by using the geometry of the genus zero moduli spaces – see [Col11b]. The general case for positive genus is not known.

Grothendieck-Teichmüller action on inertia of $\pi_1^{geom}(\mathcal{M}_{0,[n]})$. The Grothendieck-Teichmüller group $\widehat{GT}$ was first defined by V.G. Drinfel'd in the framework of quasi-Hopf quasitriangular algebras [Dri90]. Y. Ihara [Iha94] proved the existence of an injection $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$. One of the main motivations of Grothendieck-Teichmüller theory is to compare these two groups and in particular their action on fundamental groups. In this paper we compare their actions on torsion elements $\pi_1^{geom}(\mathcal{M}_{0,[n]})$.

Definition. *The group $\widehat{GT}$ is the group of the invertible elements $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{\mathbb{F}}_2'$ satisfying the equations*

(I) $\quad f(x,y)f(y,x) = 1$

(II) $\quad f(x,y)x^m f(z,x)z^m f(y,z)y^m = 1$

(III) $\quad f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51})f(x_{12},x_{23}) = 1$

where $xyz = 1$, $m = (\lambda-1)/2$ and x_{ij} are generators of $\Gamma_{0,5}$.

V. G. Drinfel'd showed that $\widehat{GT}$ acts on the profinite completion of the Artin braid groups $\widehat{B}_n$ through the formula on braid group generators σ_i

$$\sigma_i \mapsto f(y_i, \sigma_i^2)\sigma_i^\lambda f(\sigma_i^2, y_i) \qquad (\lambda, f) \in \widehat{GT}$$

where $y_i = \sigma_{i-1}\sigma_{i-2}\cdots\sigma_1^2\cdots\sigma_{i-2}\sigma_{i-1}$ – cf. [Dri90]. It is easily shown that this action passes to the quotient $\widehat{\Gamma}_{0,[n]}$ of $\widehat{B}_n$. Y. Ihara and M. Matsumoto [IM95] proved the compatibility between $\widehat{GT}$ and $G_{\mathbb{Q}}$ actions on $\widehat{\Gamma}_{0,[n]}$

Using this as well as morphisms between "flat ribbons" – cf. [LS97b], one can prove that *the action of the Grothendieck-Teichmüller group $\widehat{GT}$ on inertia generators I_D at infinity of $\mathcal{M}_{0,[n]}$ is given by λ-conjugacy.* This result means that the action of $\widehat{GT}$ on inertia at infinity is of Galois type.

This similarity with the $G_{\mathbb{Q}}$ action leads naturally to the question of whether the $\widehat{GT}$-action on profinite torsion elements has the same form. The second main result of this paper is the following.

Theorem B. *The group $\widehat{GT}$ acts on prime order torsion elements of $\widehat{\Gamma}_{0,[n]}$ by λ-conjugacy.*

In particular, since $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$, this recovers the analogous result for $G_{\mathbb{Q}}$ mentioned above, at least for prime order elements.

The first three cases of theorem B were already known since the torsion of order 2, 3 and 5 can be derived from cycles in the groups $\Gamma_{0,[4]}$ and $\Gamma_{0,[5]}$ in [LS97a]. Also by studying Galois action on specific coverings of the projective line, H. Tsunogai and H. Nakamura obtain explicit expressions on geometric torsion elements of $\Gamma_{0,[4]}$ by precising the conjugating elements – see [Tsu06], [NTY10] and [NT03]. In addition to these results, [LSN04] introduced new *geometric special loci* which are more closely related to the torsion elements, and give a fully explicit expression for the Galois action on the last case in $\Gamma_{0,[5]}$.

2. Group Cohomology, torsion properties

We follow an idea introduced by J. P. Serre in [Ser97] which serves in certain cases to relate conjugacy classes of elements of finite order of profinite completions to discrete ones. In the case of the full mapping class groups $\Gamma_{0,[n]}$ this approach can not be applied directly because these groups do not satisfy the $(\star)$ property defined below – see Example 3.5. However the methode can be adapted to work in the prime order torsion elements of $\Gamma_{0,[n]}$.

Let us consider the two following properties $(\star)$ and (H) below.

Let G be a discrete or profinite group and $\{G_i\}_I$ be a finite family of finite subgroups of G.

(1) *Every finite subgroup of G is conjugate to a subgroup of one of G_i;*
(2) *For $i \neq j$ or $g \notin G_i$ then $G_j \cap gG_ig^{-1} = \{1\}$.* $(\star)$

Remark that such a family $\{G_i\}$ must consist of exactly one representative for each conjugacy classe of the finite maximal subgroups of G. We say in what follows that *a group G satisfies the $(\star)$ property* for short, if it satisfies the $(\star)$ property for such a family.

The property (H) is defined by:

Let G be a discrete group with finite virtual cohomological dimension. Let us consider a finite family of subgroups $\{G_i\}_I$ of G. For every discrete G-module M, the morphism

$$H^n(G, M) \to \prod H^n(G_i, M) \tag{H}$$

is an isomorphism in sufficiently high degree.

In the case of a profinite group G, we adapt this (H) property by replacing discrete G-modules by torsion G-modules with continuous action.

2.1. **Good groups, properties.** We recall the notion of a good group and two well-known properties – established in a more general context by H. Nakamura in [Nak94], that are extensively used to obtain results in the next section. Recall that a group G is said to be residually finite if it injects into its profinite completion $\widehat{G}$.

Definition ([Ser94])**.** *Let G be a residually finite group and $Mod(G)$ the category of finite G-modules. The group G is said to satisfy the (A_n) property if the morphism between groups cohomology*

$$H^k(\widehat{G}, M) \to H^k(G, M) \qquad \text{for any } M \in Mod(G)$$

is an isomorphism for $k \leqslant n$ and is into for $k = n+1$. If G satisfy the (A_n) property for all $n \geqslant 1$, the group G is said to be good.

A group is said to be FP if the G-module $\mathbb{Z}$ admits a resolution of finite length by projective modules of finite type. A cohomological condition is said to be *virtual* if it is satisfied for a finite index subgroup. In particular, a FP group has finite virtual cohomological dimension.

Whereas not true in general, the profinite completion functor is left exact for good groups. The following results are well-known.

Lemma 2.1 ([Ser94, Nak94]). *Let a discrete group G be an extension of discrete groups K by H*

$$1 \to H \to G \to K \to 1$$

where H is good virtually FP and K is good. Then the following sequence is exact

$$1 \to \widehat{H} \to \widehat{G} \to \widehat{K} \to 1$$

This lemma is based on the weaker condition for G to satisfy the (A_2) property and implies the stability of the goodness property under group extension.

Proposition 2.2 ([Ser94, Nak94]). *Let a discrete group G be an extension of discrete groups K by H*

$$1 \to H \to G \to K \to 1.$$

If H is virtually FP, of finite type and good, and K is good, then G is good.

In our situation, we obtain the following results as a corollary.

Proposition 2.3. *The mapping class groups $\Gamma_{0,n}$ and $\Gamma_{0,[n]}$ are good for $n > 3$.*

Proof. Recall that for $n \geqslant 1$ the fundamental group $\pi_1(S_{g,n})$ of a Riemann surface is isomorphic to a free group, hence has cohomological dimension equal to one. Free groups are then good groups since it is sufficient to check the (A_1) property which is straightforward.

Both results are established by induction on the number of marked points. Let us consider the case of the pure mapping class group $\Gamma_{0,n}$ and remark that $\Gamma_{0,4} \simeq F_2$. Considering the Birman exact sequence coming from erasing points on surface

$$1 \to \pi_1(S_{0,n-1}) \to \Gamma_{0,n} \to \Gamma_{0,n-1} \to 1$$

where $\pi_1(S_{0,n-1}) = F_{n-2}$ is free hence good, the result is established following proposition 2.2.

The case of the full mapping class group $\Gamma_{0,[n]}$ is quite similar using the following exact sequence

$$1 \to \Gamma_{0,n} \to \Gamma_{0,[n]} \to \mathfrak{S}_n \to 1$$

where $\mathfrak{S}_n$ is good as a finite group.

□

Remark 2.4. These results can be generalized to genus 1 and 2 with similar methods: for genus 1 let us remark that $\Gamma_{1,1} = SL_2(\mathbb{Z})$ contains a congruence subgroup $\Gamma_2(3)$ which is a finite index free group; for genus 2 we use the morphism $\mathcal{M}_{2,0} \to \mathcal{M}_{2,0}/<\iota>\simeq\mathcal{M}_{0,6}$ where ι is the hyperelliptic involution. Thus the groups $\Gamma_{1,[n]}$ and $\Gamma_{2,[n]}$ are good groups. This is an important open question when $g \geqslant 3$.

The result below is based on a series of results of Serre-Huebschmann in [Hue79], K. S. Brown in [Bro94, chap. X] for discrete groups and C. Scheiderer in [Sch97] for profinite groups.

Proposition 2.5. *Let G be a discrete residually finite good group virtually FP, and $\{G_i\}_I$ be a finite family of finite subgroups of G. If G has finite virtual cohomological dimension and satisfies the $(\star)$ property, then the profinite completion $\widehat{G}$ satisfies the $(\star)$ property for the same discrete family.*

We give here a brief outline of the proof in order to explain the role of the different properties.

Sketch of proof. In the discrete case, cohomology groups $H^\star(G,\cdot)$ are isomorphic to the Farrell G-equivariant cohomology groups $\widehat{H}^\star_G(\mathcal{A}G,\cdot)$ [Bro94, chap. X] with $\widehat{H}^n_G(\mathcal{A}G,M)=\prod_I \widehat{H}^n_G(\mathcal{A}^iG,M)$ where $\widehat{H}^n_G(\mathcal{A}^iG,M)$ are cohomology groups associated to the subsimplex $\mathcal{A}^iG$ composed of subgroups conjugate to G_i. Consider the restriction morphism

$$\widehat{H}^n_G(\mathcal{A}^iG,M)\longrightarrow \widehat{H}^n_{G_i}(\mathcal{A}^iG,M)$$

where $\widehat{H}^\star_{G_i}(\mathcal{A}^iG,\cdot)$ are isomorphic to the G_i-equivariant cohomology groups $\widehat{H}^\star_{G_i}(\mathcal{A}^iG_i,\cdot)$, hence to the $H^\star(G_i,\cdot)$ groups.

If G satisfies the $(\star)$ property for the G_i, then the previous morphism is an isomorphism via an isomorphism of spectral sequences in E_1. This gives the (H) property in the case of discrete groups.

By the goodness property of G, the (H) property is transferred from G to its profinite completion $\widehat{G}$ for the same family of G_i. It is then a result of [Sch97] – based on a discrete result of [Hue79] – that profinite groups which satisfies the property (H) for some G_i satisfies the property $(\star)$ for the same G_i family. □

As a special case of the previous proposition, let us notice the following corollary for the torsion-free case, in which the family of the $(\star)$ property is reduced to $\{1\}$.

Corollary 2.6. *Let G be a discrete residually finite and good group virtually FP. If G is torsion free then its profinite completion $\widehat{G}$ is torsion free.*

2.2. **Group extension, prime order cyclic case.** We introduce in this section the theoretical framework that will be used in the mapping class group context.

Let G be a discrete group, H a torsion free subgroup of G and ρ a finite order automorphism of H induced by conjugation by a finite order element of G. Consider the discrete group $G'=H\rtimes\langle\rho\rangle\subset G$ and the set of G'-conjugacy classes of sections of

$$1\to H\to G'\to\langle\rho\rangle\to 1, \tag{1}$$

which correspond bijectively to the non-abelian cohomology set $H^1(\langle\rho\rangle,H)$.

Proposition 2.7. *Let H be a discrete torsion free group and ρ be a prime order p automorphism of H. Let us assume that the number of finite order p-cyclic conjugacy classes of $H\rtimes\langle\rho\rangle$ is finite. Then $H\rtimes\langle\rho\rangle$ satisfies property $(\star)$.*

Proof. Let us consider a family $\{G_i\}_I$ as in the remark following the property $(\star)$ in the beginning, consisting of one representative of each conjugacy classe of maximal finite subgroups of G. As H is torsion free every finite subgroup of $H \rtimes \langle\rho\rangle$ has order p, and by hypothesis there is only a finite number of G_i.

If two representatives of distinct conjugacy classes are non-trivially intersecting

$$G_i \cap gG_jg^{-1} = G_0$$

where $G_0 \neq \{1\}$, then G_0 has prime order as a subgroup of G_i. Thus $G_i = G_0$ which implies $gG_jg^{-1} = G_i$ contradicting the assumption that G_i and G_j are in two distinct conjugacy classes. □

Corollary 2.8. *Let us consider $G = H \rtimes \langle\rho\rangle$ a discrete subgroup where ρ is a prime order p automorphism of H. If G is good with a finite number of conjugacy classes of cyclic order p then $\widehat{G}$ satisfies property $(\star)$ for* the same discrete *family $\{G_i\}_I$ of G.*

Proof. The proof is straightforward as according to the previous proposition the group $G' = H \rtimes \langle\rho\rangle$ satisfies the property $(\star)$ for the discrete family $\{G_i\}_I$. Following proposition 2.5 we deduce the same property for its profinite completion. □

From a non-abelian cohomological point of view, this means that there is a bijection between the following non-abelian sets

$$H^1(\langle\rho\rangle, \widehat{H}) \simeq H^1(\langle\rho\rangle, H)$$

which are in bijection with any chosen family of discrete G_i.

Remark 2.9. Let us notice that characterising the order of the torsion of a group by cohomological properties is a well-known difficult problem. For example in the case of a discrete group G, order of torsion elements are amongst the divisors of the denominator of the Euler characteristic $\chi(G)$, but the existence of only p-torsion of G can be determined – see [Bro74].

3. The $\widehat{GT}$ action on algebraic p-torsion

This section presents the two main results of this article. Identifying the orbifold $\pi_1^{orb}(\mathcal{M}_{0,[n]})$ and geometric $\pi_1^{geom}(\mathcal{M}_{0,[n]})$ fundamental groups with the mapping class group $\Gamma_{0,[n]}$ and its profinite completion $\widehat{\Gamma}_{0,[n]}$ respectively, we apply the cohomological results of the previous section to prove theorems 3.3 and 3.10.

3.1. Geometric torsion of $\pi_1^{geom}(\mathcal{M}_{0,[n]})$.

We recall in this section some well-known results about geometric torsion of $\pi_1^{geom}(\mathcal{M}_{0,[n]})$, or equivalently about torsion element of the discrete group $\Gamma_{0,[n]}$. The following result was proved by C. MacLachlan and W. J. Harvey (cf. [MH75]).

Theorem 3.1. *Any torsion element of $\Gamma_{0,[n]}$ has order dividing n, $n-1$ or $n-2$. There exists exactly one conjugacy class in $\Gamma_{0,[n]}$ for each given order, except for order 2 and n even.*

Recall that the conjugacy class of an order k torsion elementγ of $\Gamma_{0,[n]}$ is characterised by the signature of the quotient morphism $\Gamma_{0,[n]} \to \Gamma_{0,[n]}/\langle\gamma\rangle$, which is

$$(0;\emptyset;(n-2)/k+2) \qquad (0;k;(n-1)/k+1) \quad \text{or} \quad (0;k,k;n/k).$$

For a given order $k \neq 2$, only one of these signature can be realized. For $k = 2$ and if n is even, then there exist indeed *two* conjugacy classes of order two: one for $\varepsilon_n^{n/2}$ and one for $\varepsilon_{n-2}^{(n-2)/2}$. These two conjugacy classes induce two distinct types of permutation on marked points.

Using the presentation of $\Gamma_{0,[n]}$ as a quotient of a braid group,

$$\Gamma_{0,[n]} \simeq B_n/\langle z_n = 1, y_n = 1\rangle$$

where

$$z_n = (\sigma_1\sigma_2\cdots\sigma_{n-1})^n \qquad y_n = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1^2\cdots\sigma_{n-2}\sigma_{n-1}$$

are respectively generators of the center of B_n, and the Hurwitz relation of the sphere, we obtain:

Corollary 3.2. *Every finite order element of* $\Gamma_{0,[n]}$ *is conjugate to a power of one of the following order* k *elements* ε_k *below*

$$\varepsilon_n = \sigma_1\sigma_2\cdots\sigma_{n-1} \qquad \varepsilon_{n-1} = \sigma_1\sigma_2\cdots\sigma_{n-2} \qquad \varepsilon_{n-2} = \sigma_1^2\sigma_2\cdots\sigma_{n-2}.$$

3.2. **Algebraic prime torsion of $\pi_1^{geom}(\mathcal{M}_{0,[n]})$.** We establish the first of our main results: the prime order torsion of $\pi_1^{geom}(\mathcal{M}_{0,[n]})$ is *geometric*, i.e. the conjugacy classes of such elements come from discrete ones in $\pi_1^{orb}(\mathcal{M}_{0,[n]})$.

Remark that as mapping class groups are residually finite groups, i.e. $\Gamma_{g,[n]} \hookrightarrow \widehat{\Gamma}_{g,[n]}$ then the order of the discrete torsion is preserved in the profinite completion. We note again $\varepsilon_k \in \widehat{\Gamma}_{g,[n]}$ for the images of the previous discrete torsion elements.

Let us now prove the first main result of this article.

Theorem 3.3. *Let* $\gamma \in \widehat{\Gamma}_{0,[n]}$ *be an order prime torsion element. Then* γ *is conjugate to a power of one of the elements* ε_n*,* ε_{n-1} *or* ε_{n-2}*.*

Remark 3.4. As $\Gamma_{0,4}$ is isomorphic to a free group, it follows from a result on free products that $\Gamma_{0,4}$ satisfies the $(\star)$ property (see [Ser03]).

Results of the first sections can not be directly applied[1] on $G = \Gamma_{0,[n]}$ since the mapping class groups do not satisfy property $(\star)$ in all general as noticed in the following example.

Example 3.5. Consider the mapping class group $\Gamma_{0,[30]}$. This group contains the isometry groups $\mathfrak{A}_5$ and $\mathfrak{A}_4$ of the dodecahedron and the tetrahedron respectively, as maximal finite subgroups. We notice that both groups contains an order three element – for the dodecahedron as rotation of a cube, with no marked fixed points and 10 orbits of 3 points. Therefore $\Gamma_{0,[30]}$ contains two finite maximal subgroups whose intersection is non-empty up to conjugacy.

[1]We thank B. Enriquez for this remark.

Recall that the pure mapping class groups $\Gamma_{0,n}$ are torsion free. We need the following lemma.

Lemma 3.6. *Let γ be a finite order element of $\Gamma_{0,[n]}$. Then the group $\Gamma_{0,n} \rtimes \langle\gamma\rangle$ admits a finite number of conjugacy classes of group of order p.*

Proof. Let us consider $H = \Gamma_{0,n} \rtimes \langle\gamma\rangle$ and the following exact sequence

$$1 \to \Gamma_{0,n} \to H \to \langle[\gamma]\rangle \to 1, \tag{2}$$

where $\langle[\gamma]\rangle$ is the subgroup of $\mathfrak{S}_n$ generated by the induced permutation by γ on marked points.

Then H has finite index in $\Gamma_{0,[n]}$, which implies that H has a finite number of conjugacy classes in $\Gamma_{0,[n]}$. □

We now establish that the permutation induced by any torsion element comes from a geometric one.

Lemma 3.7. *Let $\gamma \in \widehat{\Gamma}_{0,[n]}$ be a torsion element of prime order p and let us denote by $[\gamma]$ the permutation induced on marked points via the morphism $\widehat{\Gamma}_{0,[n]} \to \mathfrak{S}_n$. Then there exists an element γ_0 of the discrete group $\Gamma_{0,[n]}$ such that $[\gamma_0] = [\gamma]$.*

Proof. Let $\gamma \in \widehat{\Gamma}_{0,[n]}$ be a torsion element of prime order p, and let us consider the two following exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{\Gamma}_{0,n} & \longrightarrow & \widehat{\Gamma}_{0,[n]} & \longrightarrow & \mathfrak{S}_n & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 1 & \longrightarrow & \Gamma_{0,n} & \longrightarrow & \Gamma_{0,[n]} & \longrightarrow & \mathfrak{S}_n & \longrightarrow & 1 \end{array}$$

Suppose that $[\gamma]$ is not geometric. Let us consider the discrete group

$$H = \langle \Gamma_{0,n}, \sigma\rangle \subset \Gamma_{0,[n]}$$

where $\sigma \in \Gamma_{0,[n]}$ is an infinite order preimage of $[\gamma]$. As any permutation associated to an element of H is a power of $[\gamma]$ then H is torsion free, since if $\tau \in H$ is torsion and gives the permutation $[\gamma]^i$, then the element τ^j is torsion for $ji = 1 \bmod \mathrm{ord}([\gamma])$ and gives the permutation $[\gamma]$, which can not happen since no torsion element is associated to $[\gamma]$ by hypothesis .

Remark that the group H is good as an extension of good groups by the exact sequence (2) where $\Gamma_{0,n}$ is good according to proposition 2.2. From corollary 2.6 it then follows that the torsion-freeness of H implies the torsion-freeness of the profinite completion $\widehat{H}$. Moreover, $\widehat{H}$ is the closure of H in $\widehat{\Gamma}_{0,[n]}$, since H is of finite index and since finite index subgroups are open hence closed following [NS03], hence $\widehat{H} \subset \widehat{\Gamma}_{0,[n]}$. Now, remark that $\gamma\sigma^{-1} \in \widehat{\Gamma}_{0,n}$, hence $\gamma = h\sigma$ for $h \in \widehat{\Gamma}_{0,n}$. Thus we have $\gamma \in \widehat{H}$, which contradicts the fact that $\widehat{H}$ is torsion free.

Therefore, if $\gamma \in \widehat{\Gamma}_{0,[n]}$ is torsion, then $[\gamma]$ is geometric and thus there exists $\gamma_0 \in \Gamma_{0,[n]}$ such that $[\gamma] = [\gamma_0]$. Also, $\gamma_0^p = 1$ since $[\gamma_0^p] = 1$, so $\gamma_0^p \in \Gamma_{0,n}$ and this group is torsion free. □

We now prove theorem 3.3 by applying the cohomological results of the previous section on a geometric element associated to $[\gamma]$.

Proof of the theorem 3.3. Let $\gamma \in \widehat{\Gamma}_{0,[n]}$ be a p prime torsion element. By the previous lemma there exists a torsion element γ_0 of $\Gamma_{0,[n]}$ such that $[\gamma_0] = [\gamma]$. Since $\widehat{\Gamma}_{0,n}$ is torsion free, the preimage $\langle[\gamma]\rangle$ in $\widehat{\Gamma}_{0,[n]}$ is a semi-direct product $\widehat{\Gamma}_{0,n} \rtimes \langle\gamma\rangle$. Since γ_0 is in this preimage, there exists $h \in \widehat{\Gamma}_{0,n}$ such that

$$\gamma = h\gamma_0.$$

Let us consider the relation

$$\gamma^p = (h\gamma_0)^p = h\gamma_0(h)\cdots\gamma_0^{p-1}(h) = 1 \tag{3}$$

where γ_0 acts by conjugation on h. Then h is an order p cocycle of $H^1(\langle\gamma_0\rangle, \widehat{\Gamma}_{0,n})$. Following lemma 3.6 there exists only a finite number of conjugacy classes in the group $\Gamma_{0,n} \rtimes \langle\gamma\rangle$.

As $\gamma_0 = h^{-1}\gamma$ belongs to $\widehat{\Gamma}_{0,n} \rtimes \langle\gamma\rangle$ we have

$$\widehat{\Gamma}_{0,n} \rtimes \langle\gamma\rangle = \widehat{\Gamma}_{0,n} \rtimes \langle\gamma_0\rangle$$

Following corollary 2.8 and the fact that $\Gamma_{0,n} \rtimes \langle\gamma_0\rangle$ is good (see exact sequence (2) and argument below), the following non-abelian cohomology sets correspond bijectively

$$H^1(\langle\gamma_0\rangle, \widehat{\Gamma}_{0,n}) \simeq H^1(\langle\gamma_0\rangle, \Gamma_{0,n}).$$

Hence there exists $h_0 \in \Gamma_{0,[n]}$ which gives the same cocycle as h, i.e. the cocycles h and h_0 differ by a coboundary

$$\begin{aligned} h &= kh_0\gamma_0(k^{-1}) \\ &= kh_0\gamma_0 k^{-1}\gamma_0^{-1} \qquad\qquad \text{for } k \in \widehat{\Gamma}_{0,[n]}. \end{aligned}$$

By substitution of this relation into $\gamma = h\gamma_0$ we have

$$\gamma = k.h_0\gamma_0.k^{-1}$$

where $h_0\gamma_0$ is a torsion element of $\Gamma_{0,[n]}$ since it is a conjugate of γ.

Hence every torsion element of prime order of $\widehat{\Gamma}_{0,[n]}$ is conjugate to a geometric torsion element of same order of $\Gamma_{0,[n]}$. □

Remark 3.8. In this result and the following, restriction to prime order is only due to the Serre's theory approach we employed in section 2.2.

Remark 3.9. From this point of view, the proposition 3 of [LS94] can be seen as determining the order 2 and 3 (resp. 5) torsion in $\widehat{\Gamma}_{0,[4]}$ (resp. $\widehat{\Gamma}_{0,[5]}$).

3.3. The $\widehat{GT}$ action on torsion.

As we know that any prime order torsion element of $\widehat{\Gamma}_{0,[n]}$ is conjugate to a discrete one, we use the explicit action of the Grothendieck-Teichmüller group $\widehat{GT}$ on the braid group generators of the mapping class group $\widehat{\Gamma}_{0,[n]}$ to deduce the $\widehat{GT}$ action on prime order torsion elements of $\widehat{\Gamma}_{0,[n]}$.

Theorem 3.10. *The Grothendieck-Teichmüller group $\widehat{GT}$ acts by λ-conjugacy on the prime order torsion elements of $\widehat{\Gamma}_{0,[n]}$, i.e. if $F = (\lambda, f)$ is an element of $\widehat{GT}$ and $\varepsilon \in \widehat{\Gamma}_{0,[n]}$ a prime order torsion element, then there exists $g \in \widehat{\Gamma}_{0,[n]}$ such that*

$$F(\varepsilon) = g\varepsilon^\lambda g^{-1}.$$

The proof of this result requires us to consider a particular braid group quotient with the following property.

Proposition 3.11. *Let z_n (resp. y_n) denote the generator of the center (resp. the Hurwitz element) of the braid group B_n and let us consider the group*

$$G = B_n/\langle z_n y_n^{-1}\rangle.$$

Then G is good and every torsion element of $\widehat{G}$ is conjugate to a power of one of

$$\tilde{\varepsilon}_{n-2} = \sigma_1^2\sigma_2\cdots\sigma_{n-2} \qquad \tilde{\varepsilon}_{n-1} = \sigma_1\sigma_2\cdots\sigma_{n-2} \quad \text{or} \quad \tilde{\varepsilon}_n = \sigma_1\sigma_2\cdots\sigma_{n-1}$$

respectively of order $n-2$, $n-1$ and n in G.

Proof. Let us show that G is good. We consider the exact sequence

$$1 \longrightarrow Ker(\psi) \longrightarrow G = B_n/\langle y_n^{-1}z_n\rangle \xrightarrow{\psi} \Gamma_{0,[n]} = B_n/ < y_n, z_n > \longrightarrow 1 \tag{4}$$

and let us identify $Ker(\psi)$ with $\mathbb{Z}$ using the morphism $\phi : B_n \to G$ where we note $\bar{y}_n = \phi(y_n)$ and $\bar{z}_n = \phi(z_n)$. The element $\bar{y}_n$ is central as $\bar{y}_n = \bar{z}_n$ in G. Hence for $\omega \in Ker(\psi)$ we have $\omega = \bar{y}_n^j\bar{z}_n^k = \bar{z}_n^{j+k}$ in G and $Ker(\psi)$ is cyclic.

The group G is thus good, of finite type and FP group extension and we conclude using the proposition 2.2.

Let γ be a torsion element of $\widehat{G}$ of prime order p. Since G is good and of finite type the profinite completion functor is exact according to lemma 2.1. The exact sequence (4) then induces

$$1 \longrightarrow Ker(\widehat{\psi}) \longrightarrow \widehat{G} \xrightarrow{\widehat{\psi}} \widehat{\Gamma}_{0,[n]} \longrightarrow 1 \tag{5}$$

where $Ker(\widehat{\psi}) \simeq \widehat{\mathbb{Z}}$, hence torsion free – or more generally by proposition 2.6. Considering $\widehat{\psi}(\gamma) \in \widehat{\Gamma}_{0,[n]}$, then $\widehat{\psi}(\gamma)$ and γ have the same order.

Following proposition 3.3 the element $\widehat{\psi}(\gamma)$ is conjugate to a power of a geometric element of $\Gamma_{0,[n]}$

$$\psi(\gamma) = g\varepsilon_i^k g^{-1} \text{ with } g \in \widehat{\Gamma}_{0,[n]} \text{ and } i \in \{n-2, n-1, n\}.$$

As $\tilde{\varepsilon}_i$ is a preimage in $\widehat{G}$ of ε_i we obtain

$$\gamma = \tilde{g}\tilde{\varepsilon}_i^r\tilde{g}^{-1}\bar{z}_n^j \tag{6}$$

where $\bar{z}_n$ is the generator of $Ker(\widehat{\psi})$ and $\tilde{g}$ is a preimage of g in G. Powering the relation above to the order p of γ we obtain

$$\bar{z}_n^{pj} = 1$$

as z_n generates the center of $\widehat{B}_n$. This implies $j = 0$ since $Ker(\psi) \simeq \widehat{\mathbb{Z}}$ is torsion free as a group of cohomological dimension one. We conclude with equation (6) that γ is conjugate to a geometric element in $\widehat{G}$ since

$$\gamma = \tilde{g}\tilde{\varepsilon}_i^r\tilde{g}^{-1}$$

with $i \in \{n-2, n-1, n\}$ as announced. □

Lemma 3.12. *The $\widehat{GT}$ action defined on the braid group $\widehat{B}_n$ factors through the quotient $\widehat{G} = \widehat{B}_n/\langle z_n y_n^{-1}\rangle$. This action is compatible with the morphism $\widehat{G} \to \widehat{\Gamma}_{0,[n]}$.*

Proof. The proof is straightforward and comes from the action on $\widehat{B}_n$ defined in [Dri90]. For $F \in \widehat{GT}$ we have

$$F(y_n) = y_n^{\lambda} \qquad F(z_n) = z_n^{\lambda}.$$

Hence $F(z_n y_n^{-1}) = (z_n y_n^{-1})^{\lambda}$ since z_n generates the center of $\widehat{B}_n$. Then the action of $\widehat{GT}$ on $\widehat{B}_n$ and $\widehat{\Gamma}_{0,[n]}$, defined on each generator σ_i, is defined on a compatible manner on $\widehat{G}$

$$\begin{array}{ccc} \widehat{B}_n & \longrightarrow & \widehat{\Gamma}_{0,[n]} \\ & \searrow & \uparrow \\ & & \widehat{G} = \widehat{B}_n/\langle z_n y_n^{-1}\rangle \end{array}$$

hence respects the morphism $\widehat{G} \to \widehat{\Gamma}_{0,[n]}$. □

We now prove the main theorem of this article.

Proof of theorem 3.10. Let γ be a p-prime order torsion element of $\widehat{\Gamma}_{0,[n]}$ and $F = (\lambda, f)$ be an element of $\widehat{GT}$. Following proposition 3.3 the element γ is conjugate to a r-power for some $r \in \mathbb{Z}$ of one of the maximal finite order element ε_n, ε_{n-1} or ε_{n-2}.

Since $F(\gamma)$ has same order as γ, it is conjugate to a power of one of these elements according to the same proposition. As the group $\widehat{GT}$ preserves permutation and as each of the distinct elements ε_k has a distinct number of fixed points, we deduce

$$F(\gamma) = \alpha\gamma^k\alpha^{-1}$$

for some $k \in \widehat{\mathbb{Z}}$. Let us consider the commutative diagram

$$\begin{array}{ccc} \widehat{\Gamma}_{0,[n]} & \longrightarrow & \widehat{\Gamma}^{ab}_{0,[n]} \\ \downarrow F & & \downarrow F \\ \widehat{\Gamma}_{0,[n]} & \longrightarrow & \widehat{\Gamma}^{ab}_{0,[n]} \end{array}$$

which gives the relation

$$F(\gamma^{ab}) = F(\gamma)^{ab}. \tag{7}$$

We now compute in the abelianisation of $\widehat{\Gamma}_{0,[n]}$ to determine the k power. Since $\widehat{\Gamma}_{0,[n]} = B_n/\langle z_n, y_n\rangle$, in $\widehat{\Gamma}^{ab}_{0,[n]}$ one obtains $\sigma_i = \sigma$ for $1 \leqslant i \leqslant n-1$ from braid relation, $\sigma^{n(n-1)} = 1$ from relation $z_n = 1$ and $\sigma^{2(n-1)} = 1$ from relation $y_n = 1$. Hence

$$\widehat{\Gamma}^{ab}_{0,[n]} \simeq \begin{cases} \mathbb{Z}/(n-1) \text{ if } n \text{ is odd} \\ \mathbb{Z}/2(n-1) \text{ if } n \text{ is even.} \end{cases}$$

Let us first consider the case $\gamma \sim \varepsilon_{n-1}^r$. Then from the expression $\varepsilon_{n-1} = \sigma_1 \dots \sigma_{n-2}$ in corollary 3.2 and theorem 3.3, the equation (7) gives

$$\sigma^{r(n-2)\lambda} = \sigma^{r(n-2)k}$$

and computing modulo the order of the element γ

$$(n-2)\lambda \equiv (n-2)k \mod p$$
$$\lambda \equiv k \mod p$$

since p divides $n-1$.

For the case $\gamma \sim \varepsilon_n^r$ in $\widehat{\Gamma}_{0,[n]}$, let us consider the subgroup $\widehat{\Gamma}^1_{0,[n+1]}$ in $\widehat{\Gamma}_{0,[n+1]}$ constituted by elements fixing a point, and the orbifold fundamental groups homomorphism induced by erasing the fixed point

$$\widehat{\Gamma}^1_{0,[n+1]} \to \widehat{\Gamma}_{0,[n]}.$$

Then $\gamma = (\sigma_1 \cdots \sigma_{n-1})^r \in \widehat{\Gamma}^1_{0,[n+1]}$ is sent to ε_n^r which is of order n in $\widehat{\Gamma}_{0,[n]}$. Thus the $\widehat{GT}$ action on γ is induced by its action on ε_n.

The last case $\gamma \sim \varepsilon_{n-2}^r$ is more subtle since $\gamma^{ab} = \pm 1$ in $\widehat{\Gamma}^{ab}_{0,[n]}$. We then consider the quotient $\widehat{G} = \widehat{B}_n/\langle z_n y_n^{-1}\rangle$ in the braid group through the factorization

$$\begin{array}{ccc} B_n & \longrightarrow & \Gamma_{0,[n]} \\ & \searrow & \uparrow \\ & & B_n/\langle z_n y_n^{-1}\rangle \end{array}$$

whose abelianisation is isomorphic to

$$\widehat{G}^{ab} \simeq \widehat{\mathbb{Z}}/(n-1)(n-2).$$

Following proposition 3.11 let us consider $\tilde{\varepsilon}_{n-2}$ the obvious preimage of ε_{n-2} in $\widehat{G}$. We remark that its abelianisation $\tilde{\varepsilon}^{ab}_{n-2}$ has same order as ε_{n-2} in $\widehat{G}^{ab}$.

Let $\tilde{\gamma}$ be a preimage of γ in $\widehat{G}$ and let $F \in \widehat{GT}$. Then $\tilde{\gamma}$ is conjugate to a geometric element in G by proposition 3.11 and we can consider the action of F on $\tilde{\gamma}$ by lemma 3.12. Because F preserves permutations, $F(\tilde{\gamma})$ is conjugate to a power of the same geometric element as γ, so we have:

$$F(\tilde{\gamma}) = \beta^{-1}\tilde{\gamma}^m\beta \text{ with } \beta \in \widehat{G}. \tag{8}$$

Computing in the abelianised $\widehat{G}^{ab}$ as previously, since the $\widehat{GT}$ action on $\widehat{G}$ commutes with abelianisation, the equations

$$F(\tilde{\gamma})^{ab} = F(\tilde{\gamma}^{ab})$$

and (8) imply

$$g^{r\lambda(n-1)} = g^{rm(n-1)}$$

where g is a generator of $\widehat{G}^{ab}$. As $g^{r(n-1)}$ has same order p as $\tilde{\gamma}$ we have $\lambda = m \ mod \ p$. Thus

$$F(\tilde{\gamma}) = \beta^{-1}\tilde{\gamma}^{\lambda}\beta \quad \text{where } \beta \in \widehat{G}.$$

Following lemma 3.12 this relation is the same in $\widehat{\Gamma}_{0,[n]}$. □

Due to the fact that elements considered have prime order, the computations in the proof could be simplified. However remark that the proof given above is adapted to work for any order *as soon as* it is known that the action of an element of $\widehat{GT}$ preserves conjugacy classes of groups generated by profinite torsion elements.

For example the exact analogous result can be established for the absolute Galois group $G_{\mathbb{Q}} \subset \widehat{GT}$ acting on *any geometric torsion element of* $\pi_1^{geom}(\mathcal{M}_{0,[n]})$, *regardless of order*. The reason is that an explicit description of the *the special loci* – defined as the substacks of $\mathcal{M}_{0,[n]}$ whose closed points admit the torsion element as automorphism – cf. [Sch03], is used to control the action of $G_{\mathbb{Q}}$ on the conjugacy classes – cf. [Col11b].

In the case of genus one, a complete description of special loci is not available, hence the Galois action on the corresponding geometric inertia is not fully determined. Moreover, it is not known whether $\widehat{GT}$ acts on the *full mapping class groups* $\widehat{\Gamma}_{g,[n]}$ for $g \geqslant 1$.

In [Col11a] we define a new Grothendieck-Teichmüller group $\widehat{GR}$, defined in the torsion context of [Sch06], which contains $G_{\mathbb{Q}}$ and acts on the full mapping class groups $\widehat{\Gamma}_{g,[n]}$. The cohomological results of section 1 on prime order extension of profinite torsion free groups are still usable. We adapt them to a description of discrete torsion conjugacy classes of $\Gamma_{1,[n]}$ to obtain analogous results for Grothendieck-Teichmüller action on genus one profinite torsion elements.

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