Anabelian Arithmetic Geometry - A new Geometry of Forms and Numbers: Inter-universal Teichmüller theory or "beyond Grothendieck's vision"

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This text presents an informal overview on how, in accordance with some deeply rooted principles of the philosophy of Alexander Grothendieck concerning the practice of mathematics, recent progress in anabelian arithmetic geometry has led to the inter-universal Teichmüller theory (IUT) of Mochizuki Shinichi. The new geometry of monoids furnished by IUT may be understood as the result of a seminal encounter between Grothendieck's principle of resolving the tension between the discrete and continuous realms, on the one hand, and p-adic Hodge theory and height theory, on the other. In doing so, it opens a new research frontier that goes beyond the Grothendieck geometry of rings-schemes by providing a unifying framework for Diophantine and anabelian arithmetic geometry.

Keywords: *Arithmetic homotopy and Galois theory, anabelian geometry, Diophantine geometry, Inter-universal Teichmüller geometry, Galois-Teichmüller theory, abc and Vojta Conjectures, Fermat's Last Theorem.*
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"[...] nothing seems to happen, and yet at the end a highly non-trivial theorem is there."
– P. Deligne on Grothendieck's "Rising Sea" method, in [Del98].

A Panorama of Grothendieck's Anabelian Philosophy

At the most elementary level of human cognition, the essence of anabelian arithmetic geometry concerns the articulation – and possible reconciliation – of two *a priori* complementary ways of deciphering the perceptible world, that is to say, the discrete and continuous realms. The discreteness of number theory – or its *geometric variation*, Diophantine geometry – imposes constraints on the transcendental nature of the mind, while the continuous nature of geometry – or its *discrete variation*, arithmetic geometry – furnishes a comfortable receptacle for its various realizations.

§ Grothendieck's Mathematics Philosophy. In response to the limitations of the human mind to grasp the essence of the dichotomy between the discrete and the continuous, Grothendieck's approach is to propose a universal and structural vision that reconciles the two realms (the discrete and the continuous) – each as an avatar of a unique functorial and category-theoretic construction, see Fig. 1 below and [R&S] more generally. This approach is both philosophical and practical: *languages are constructed that reveal pre-existing structures*, which, in turn, stimulate the further development of language[1]. In Grothendieck's twelve themes legacy, the unified treatment of Galois symmetries of numbers and geometric forms – i.e., *anabelian geometry and Galois-Teichmüller theory* – is presented as a "master theme" to the mathematical community, see Fig. 2 and ibid.

One can consider that the new geometry *is, before anything else, a synthesis between these two worlds, until then adjoining and closely interdependent, but nevertheless separated: the "arithmetic" world, in which live the (so-called) "spaces" without principle of continuity, and the world of the continuous magnitude, where live the "spaces" in the proper sense of the term, accessible to the means of the analyst [...]. In the new vision, these two worlds, formerly separated, form only one [..., in the]* vision of an "arithmetic geometry" *(as I propose to call this new geometry).*[A]

Fig. 1. *A. Grothendieck in "Récoltes et semailles" (1986),* § The new geometry - or the marriage of number and grandeur *[R&S].*

The most profound (in my eyes) of these twelve themes [or "master themes" of my work], are the theme of motives, and the closely related theme of anabelian algebraic geometry and the Galois-Teichmüller yoga.[B]

Fig. 2. *A. Grothendieck in "Récoltes et semailles" (1986), notes 22 & 23 [R&S].*

The resolution of this discrete-continuous tension between number theory and geometry is what constitutes the core of anabelian arithmetic geometry. This process not only involves multiple areas of mathematics – including class field theory, low-dimensional topology, topological group theory, complex algebraic geometry and analytic Teichmüller geometry, but also requires a dedicated effort to acquire a specific way of mathematical thinking[2].

§ "Esquisse" of a Fruitful International Legacy. When Grothendieck retired from the mathematical community, he left his peers a stimulating research legacy that includes his anabelian vision: the structure is given, but the objects have yet to be defined. In his 1983 "Letter to Faltings" [Gro97], a first introduction to the anabelian yoga is sketched, that was later developed further as a broader "Galois-Teichmüller theory" in his 1.600 page private manuscript the "Long March through

[1] The philosophical aspects of Grothendieck's practice of mathematics, i.e., more specifically, the creation of unifying and revealing mathematical contexts are discussed in [R&S] § 2.8-"La vision - ou douze thèmes pour une harmonie". As discussed in a recent conference at Chapman University [Chap23], this approach could serve as a virtuous guide for the modern arithmetic geometer. Regarding Grothendieck's approach, where "language is invented and structures are discovered", see Panza's talk ibid.

[2] This constant discrete-continuous tension – in addition to the use of a language-formalism-picture triangle and, following Grothendieck's philosophy, the introduction of a rich and necessary terminology – is what makes anabelian geometry of a total different nature – and thus difficult to grasp – even for specialists in closely related fields, such as number theorists or non-anabelian algebraic geometers.

Galois theory" [LMG]. This research finally culminated in his 1984 "Esquisse d'un programme" [Esq], that inspired the realization of an international research program[3] (~1990-2010).

What began as multiple isolated research directions in the international community[4] – for example, in France with the work of Lochak and Schneps on Grothendieck-Teichmüller theory, in Germany then in the US with the work of Pop on anabelian birational geometry and the work of Fried on Regular Inverse Galois theory, and in Japan with the independent development of Ihara's program – resulted in the 90's in *unified international research efforts* with multiple breakthroughs and long-term collaborations. In Japan, the first generation – which centered around the school of Ihara Yasutaka and Oda Takayuki – was followed by a second generation – constituted by Matsumoto Makoto, Nakamura Hiroaki, Tamagawa Akio, and Mochizuki Shinichi. The school of Mochizuki Shinichi, in turn, produced a third generation – namely, Hoshi Yuichiro – and, more recently, a fourth generation of junior researchers. All of these developments in Japan centered around the "Research Institute for Mathematical Sciences", Kyoto University, Japan (RIMS), which is now the only international mathematics institute that has preserved, vastly expanded and renewed this unique anabelian culture[5].

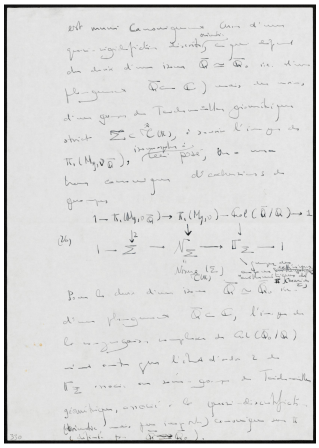


Fig. 3. *La "Longue Marche" à travers la théorie de Galois.*

§ A "far from abelian" Galois Arithmetic Geometry. The focus of anabelian arithmetic geometry, which encompasses both anabelian and Galois-Teichmüller geometry, is to explore the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of the rational number field – the seed of number theory that controls all the Galois symmetries that arise from the rational numbers – in the framework of Grothendieck's theory of the étale fundamental group, the group $\pi_1^{\mathrm{et}}(X)$ of paths or loops on a space X, see Fig. 6. In this category-theoretic unifying context, *the Galois group and the group of paths are two avatars of a unique construction* that unifies the discreteness of the former and the continuity of the latter.

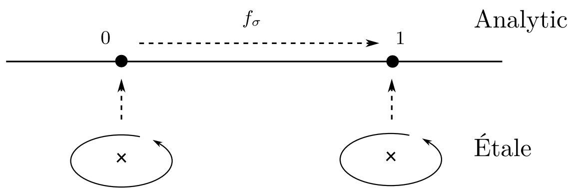


Fig. 4. ***Grothendieck-Teichmüller theory*** *is a group-theoretic, combinatorial approach to the absolute Galois group of the field of rational numbers, which relies on* an étale-analytic transport: *on the sphere* $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$*, an analytic isomorphism transports the étale monodromy loop around* 0 *to the one around* 1 *for an essential arithmetic invariant* f_σ *to appear.*

Contrary to another theme of Grothendieck's arithmetic geometry – linear Galois representation theory, which was used by Grothendieck (1965), then Deligne (1974) to establish the Weil conjectures – and to the modularity program for elliptic curves – which leads, via Taniyama–Shimura–Weil and Ribet's Theorem, to Wiles' proof of Fermat's Last Theorem (1994) – *anabelian arithmetic geometry is far from being linear and abelian*, i.e., based on a commutative fundamental group. The natural domain of anabelian arithmetic geometry is that of hyperbolic curves of a given genus g with m

[3] In the Russian mathematical community, where the "Esquisse" was distributed as an underground "Samizdat", it should be no surprise that George Shabat and Vladimir Voevodsky – first in terms of "Dessins d'enfants" [SV90], then in terms of anabelian results [Voe91] – were the first to initiate a systemic study of Grothendieck's legacy, see [Sha18]. It is worth remembering that the theory of "Dessins" indeed originates from a famous result of Belyĭ – and a remark by Bogomolov – whose announcement made a strong impression on Grothendieck[C]. Anabelian geometry over algebraic closed fields was later developed by Bogomolov and Tschinkel.

[4] All these approaches share, in terms of their common Grothendieckian generating process and origins, some similarity with the anabelian geometry that is discussed in these notes.

[5] It may be of interest to note that Japan was the first country in which "Récoltes et Semailles" was published, with the authorization of Grothendieck, namely, by Gendai Sugakusha Ed., in 1989. It was translated into Japanese by Tsuji Yuichi.

marked points – see Fig. 6 – or their moduli spaces $\mathcal{M}_{g,[m]}$, which classify the families, deformations, and internal symmetries of such curves.

In terms of mathematical objects – following the realization of "Esquisse" given by Ihara, Lochak, Matsumoto, Nakamura, and Schneps, and, as well as the realization given by Drinfel'd via quantum group theory – anabelian geometry involves Teichmüller spaces (i.e., the analytic deformation of complex structures), mapping class groups and braid groups (where braid crossings are non-commutatively composed), topological group theory, and, more recently, the theory of operads. It indeed *thrives on seminal encounters* – e.g., with Deligne's theory of weights, Thurston's progress in Teichmüller theory – from which it borrows seminal insight and techniques.

§ In Japan beyond the Grothendieckian Vision. The Japanese school achieved decisive progress in the anabelian program for curves, namely, with the work of Nakamura – via Deligne's theory of weights – then with the work of Tamagawa – via class field theory and the Lefschetz trace formula to detect rational points on covers – which also includes an anabelian Néron–Ogg–Shafarevich–Serre–Tate good reduction criterion. The Japanese school went further to produce higher dimensional anabelian results for configuration spaces. The zero dimensional case, that is to say, the reconstruction of a number field from its absolute Galois group, follows from previous work by Neukirch-Uchida (~1977).

The next breakthrough, which includes several substantial strengthenings of Grothendieck's original anabelian conjecture for hyperbolic curves over number fields, came from a decisive shift of perspective by Mochizuki Shinichi (1995), as the result of an encounter with Faltings' p-adic Hodge theory, *from working with spaces over number fields $K/\mathbb{Q}$ to working with spaces over p-adic local fields $K/\mathbb{Q}_p$*[6], i.e., over formal neighborhoods around a given prime p. This shift from Grothendieck's original global vision to working over local fields – which gives rise to stronger results that typically include the global number field case – has since proven to be the most natural anabelian setting and has yielded a plethora of progress (see [Hos22]), which also stimulated further work on other variants of anabelian geometry[7].

Mochizuki's anabelian breakthrough has, with the support of the Japanese anabelian community, since ramified into some additional *innovative absolute, mono-, and combinatorial anabelian variants*, each elucidating further the essential nature of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. The final reconciliation of the continuous and discrete realms – i.e., *anabelian arithmetic geometry and the Diophantine geometry of estimates* – however required another decisive step[8]. The realization of this missing link – which may be regarded as the final chapter of a 20-year long personal journey and indeed harks back to the mathematical and philosophical legacy of Grothendieck – was provided by Mochizuki's *inter-universal Teichmüller theory* in 2012 – for a survey of the theory in its author's own words, we recommend [Alien] and the more recent [EssLgc]; for the general scientist, see [Fes16]. The new geometry furnished by IUT opens new horizons that lie beyond Grothendieck's original algebraic geometry and the realm of schemes, rings, and fields – structures with two operations – by working in the *more flexible realm of multiplicative and additive monoids* – structures with only one operation – and has already yielded some decisive and central results in number theory (see § An Anabelian abc Inequality), as well as important breakthroughs related to the original framework of Galois-Teichmüller theory.

※ *Note to the reader: This text is intended for mathematically oriented scientists or curious mathematics students. For their convenience, special care has been given to include a reasonable use of mathematical notations. Such notations should be considered as anchors for the mind and as concrete bridges to the original texts and cited manuscripts.*

6 For further details, we refer to the always stimulating survey of Nakamura-Tamagawa-Mochizuki [NTM98].

7 One could cite the "close-to-abelian" program of Nakamura and Tamagawa for curves, of Tamagawa and Saïdi for fields, and of Pop and Topaz for function fields (indeed originally Bogomolov's 1990 program).

8 At this stage, it is interesting to note that, originally and for a long time, the bridge between anabelian and Diophantine geometry was expected to result from an argument to the effect that the Grothendieck Section Conjecture implies Mordell's Conjecture; we refer to "Letter from Deligne to Thakur" in [Sti11] and Remark 9 ibid. for a counter-argument.

Acknowledgements. This manuscript should be understood as the outcome of multiple collaborative efforts within the arithmetic anabelian community. It is the initial study of IUT by *Yamashita Go*, *Mohammed Saïdi*, and *Hoshi Yuichiro*, together with the work of the speakers at the 2015 Oxford and 2016 Kyoto conferences, that have led to what is today's general understanding of IUT. At a more personal level, the author would like to express his gratitude to the speakers and participants of the "Promenade in Inter-universal Teichmüller theory" 2020-2021 RIMS-Lille seminar and the "Expanding Horizons of Inter-universal Teichmüller Theory" 2021 workshop series, and especially to *Mochizuki Shinichi*, *Ivan Fesenko*, *Pierre Dèbes*, *Hoshi Yuichiro*, *Emmanuel Lepage*, *Tsujimura Shota*, and *Minamide Arata* for regular discussions, and to *Dinesh Thakur* for his attentive proofreading. This work was supported by the International Center for Next-Generation Geometry, a center affiliated with the *Research Institute for Mathematical Sciences* located in Kyoto University.

The Discrete-continuous Frontier

Diophantine and anabelian arithmetic geometry, while both relying on Grothendieck's geometry of schemes, are of an essentially different nature. The motivating force behind Diophantine geometry consists, for the most part, of questions concerning properties of *numbers within geometry* such as the existence of rational points in varieties and questions of estimates, see Fig. 5, where *"the geometry governs the arithmetic"*. Anabelian arithmetic geometry, on the other hand, reflects a more symbiotic relationship between the *symmetry properties of numbers and spaces* in terms of Galois groups and étale fundamental groups – or groups of paths, as in Fig. 6.

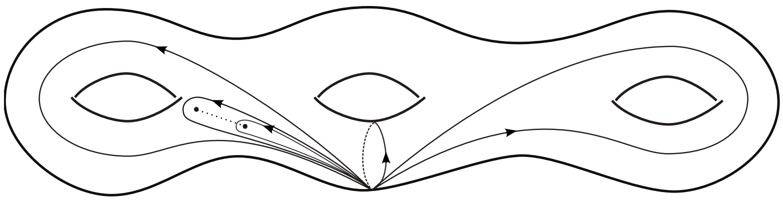
Fig. 5. ***Thue-Siegel-Roth Theorem*** *on bad rational approximation of algebraic numbers.*
Given $\alpha \in \bar{\mathbb{Q}}$ an algebraic number, then for any $\varepsilon > 0$, there exist only finitely many $p/q \in \mathbb{Q}$ *such that*

$$|\alpha - p/q| < 1/q^{2+\varepsilon}.$$

While the anabelian realm is dominated by a unique reconstruction goal, the Diophantine realm is speckled with a vast intricate network of inter-related conjectures. Among them:

- *Oesterlé-Masser's abc conjecture (1985)* – or in its original geometric version, the Szpiro conjecture for elliptic curves (1981) – is a seminal conjecture at the center of this network;
- *Mordell's conjecture (1922)* asserts *the finiteness of the set of rational points on an algebraic curve of any genus* – proven by Faltings (1983) – and is the first decisive application of algebraic geometry to height theory;
- *Vojta's conjecture (1987)* provides an innovative and fruitful geometric insight into Diophantine problems.

Fig. 6. ***An anabelian hyperbolic curve.***



Anabelian geometry *deals with the reconstruction of a curve from its group of paths, as depicted in this geometric picture of a genus 3 m-pointed curve, where each loop corresponds to, in the arithmetic case, a "rotational" cyclotome $\widehat{\mathbb{Z}}(1)$. Abstract group-theoretic synchronizations between the loops corresponding to cyclotomes and the disentanglement of their arithmetic and geometric properties constitute essential steps in many anabelian reconstruction algorithms.*

Anabelian arithmetic geometry, on the other hand, is already based on an *essential unification of number theory and geometry*, since Grothendieck's arithmetic fundamental group $\pi_1^{\text{et}}(X)$ may be identified either with $\widehat{\pi}_1^{\text{top}}(X)$ – a profinite version of the group of paths on the underlying topological space of X when X is a complex algebro-geometric space – or with the absolute Galois group $\text{Gal}(\bar{k}/k)$ when $X = \text{Spec}\, k$. The étale fundamental group of a stack or scheme X over a number field[9] k appears furthermore as an extension of $\widehat{\pi}_1^{\text{top}}(X)$ by $\text{Gal}(\bar{k}/k)$ – see § Anabelian Reconstructions - Schemes and Monoids. The theory has evolved into different variants – relative/absolute, bi/mono, combinatorial – each of which sheds light on certain aspects of the number-geometry intertwining.

Number Theory, Diophantine Geometry & Arithmetization. The roots of modern Diophantine geometry may be traced back to the purely number-theoretic work of Thue-Siegel (see

9 Strictly speaking, embedded in the field of complex numbers.

Fig. 5), as well as to the function field-theoretic insight of Hilbert, Kronecker, then Mordell and Siegel. On the other hand, the geometrization of Diophantine problems received substantial impetus from the efforts of Lang in the 1960's, who not only coined the term "Diophantine geometry", but also gave structure to the field by proposing numerous – now famous – conjectures that embodied insights from analytic and complex geometry and, ultimately, Grothendieck's algebraic geometry[10].

From these efforts of Lang resulted two of the most central conjectures of number theory and Diophantine geometry: *Masser-Oesterlé's abc conjecture* – whose numerous variants include many conjectures of Diophantine geometry, and which we present in its non-geometric (and non-Szpiro) form for the reader's convenience – and *Vojta's conjecture* on heights of points of varieties[11], which, in the case of the abc Conjecture, constitutes a sort of quintessential example of the geometrization of Diophantine estimates.

The shift from algebraic geometry to *Galois-theoretic arithmetic geometry* has its origins in Mochizuki's [GenEll] – which, at the time (2009), went largely unnoticed – where a link is established between Diophantine conjectures, on the one hand, and the introduction of powerful methods from Faltings' proof of the Mordell Conjecture[12], on the other. This shift may be seen in the various Galois-theoretic aspects of the *"dilation"* of elliptic curves, which ultimately developed into one of the core ideas underlying IUT geometry.

※ *For an accessible, detailed, and up-to-date presentation of Diophantine geometry and height estimate theory, we refer the reader to Bombieri-Gubler's book [BG06].*

§ "abc" - The Shadow of a Network of Conjectures. The abc Conjecture, proposed by Masser and Oesterlé, asserts, in its essence, *a deep and subtle relationship between the addition and multiplication of integers:*

> **The abc Conjecture (1985).** For every $\varepsilon > 0$, there exists a constant K_ε such that any triple of integers (a, b, c) coprime such that $c = a + b$ satisfies
>
> $$c \leqslant K_\varepsilon \cdot \mathrm{rad}(abc)^{1+\varepsilon},$$
>
> where $\mathrm{rad}(n)$ denotes the product of prime divisors of n.

– or in Waldschmidt's words: *when two numbers a and b are divisible by large powers of small primes, $a + b$ tends to be divisible by small powers of large primes.* The abc Conjecture can be seen as a bound on the difference between the additive and multiplicative structures of the rational numbers, see also § An Anabelian abc Inequality.

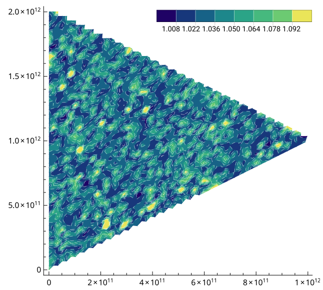


Fig. 7. ***An*** (a, b) ***description of large*** $\sim 10^{12}$ ***abc-triples.***

Despite its elementary formulation, the abc Conjecture seems at first glance beyond reach. Various computational attempts to identify properties of abc triples, including in terms of elliptic curves, *did not provide any new insight into the conjecture* – see for example Fig. 7, where colors are used to indicate the $(1 + \varepsilon)$-quality of an abc-triple.

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10 On the attitude of Lang toward the incorporation of Grothendieck's developing and at that time already colossal *Éléments de géométrie algébrique*, we refer to [Lan95]. This text, whose polemical tone does not diminish its contemporary pertinence, indeed presents two interesting examples of social obstacles that may hinder the circulation of new mathematical ideas – *"If Algebraic Geometry really consists of (at least) 13 Chapters, 2.000 pages, all of commutative algebra, then why not just give up? The answer is obvious."* (ibid.).

11 A height function measures the size of rational points on varieties, both *locally* around primes and *globally* on global number fields. An Arakelov-theoretic variant provides an even more functorial approach in terms of arithmetic line bundles as in the context of Faltings' and Mochizuki's approaches, see § Inter-universal Teichmüller Geometry.

12 While Faltings' proof of the Mordell Conjecture constitutes a groundbreaking encounter between number theory and geometry, the arithmetic of elliptic curves is however not yet fully revealed in Faltings' work. We prefer to stress its seminal role in Mochizuki's anabelian geometry breakthrough, see Fig. 10 and the corresponding section.

The abc Conjecture is deeply related, via its various variants, to some of the most central conjectures of analytic number theory, Diophantine geometry and arithmetic geometry. For example, abc implies (* denotes equivalences):

- the Mordell Conjecture (Elkies 1991);
- the Thue-Siegel-Roth Theorem of Fig. 5 (Bombieri 1994);
- no Siegel zeros for Dirichlet L-functions (Granville-Stark 1999);
- the (strong) Hall Conjecture*;
- the (generalized) Szpiro Conjecture*;

where the Szpiro Conjecture, via consideration of Frey-Hellegouarch curves $y^2 = (x-a)(x-c)$ for (a,b,c) such that $a^n + b^n = c^n$, implies Fermat's Last Theorem above a certain degree[13].

It was the brilliant insight of Vojta to reformulate abc as a geometric Diophantine inequality in terms of heights and ramification, by considering an (a,b,c) triple as a point $[a:b:c] \in \mathbb{P}^1 \setminus \{0,1,\infty\}$ on a genus 0 curve, or, alternatively, the projective line, see Fig. 8. While this new geometric insight made precise a shadow area that had yet to be explored, no conceptual progress was made for 20 years, thus illustrating the limits of traditional arithmetic thinking and *suggesting the need for a profound change of prism in arithmetic geometry.*

§ From Vojta to Mochizuki - Geometrization & Arithmetization. As expressed by Dèbes, "the Vojta conjecture appears as a universal geometric Diophantine Approximation statement – rational points are rare because algebraic numbers are badly approximated by rational numbers in the sense of [the] Thue-Siegel-Roth Theorem".

Fig. 8. ***The Vojta Conjecture (1987).*** *For X a variety, K a number field, S a finite set of primes of K, $D < X$ a divisor with normal crossings, H an ample divisor, and $\varepsilon > 0$:*

$$m_{S,D}(P) + h_{K_X}(P) \leqslant \varepsilon \cdot h_H(P) + O(1)$$

for every point P of X outside of some (proper) subvariety of X.

The Vojta Conjecture*, which was inspired by Nevanlinna theory in complex analysis, is a geometrization of Thue's inequality: D can be seen as a subvariety of X, the function $m_{S,D}$ evaluates the* proximity of the point to *D locally at the primes in S, see also (LHS) of Fig. 5; each height function $h_\bullet$ evaluates the size of P in terms of a certain referential system of coordinates describing X that arises from the divisor "$\bullet$" – see (RHS) of Fig. 5 with $x_P = q$ as the coordinate of a point P on the line. We refer to [BG06] Conj. 14.3.2 for further details and ibid. Conj. 14.3.11 for the ramification version.*

The Vojta Conjecture is an essential step in introducing geometric insight into the world of rational numbers and estimates: the *abc inequality now appears as a bound on the height of a point on a curve by means of the ramification of the coordinates of the point*[14]; in this form, the bound on the height generalizes naturally *to higher dimensions.* In the case where $X = \mathbb{P}^N$ is N-dimensional and D is a union of (linear) hyperplanes, the Vojta Conjecture translates into Schmidt's Subspace Theorem (1972), which for $N = 1$ corresponds to the Thue-Siegel-Roth Theorem of Fig. 5[15]. In order to illustrate the intricacy of this circle of ideas, it may be interesting to note that Vojta also provided a new proof of the Mordell Conjecture – now simplified by Bombieri using the Thue-Siegel-Roth Theorem and the theory of heights – that leads to further and rich generalizations. We refer to Rémond's survey [Rém03] for more details.

The decisive step that allows one to go *beyond Vojta's geometrization of the abc Conjecture* – and which also paves the way for the introduction of anabelian geometry and the subsequent development

13 Interestingly, the polynomial analogue of abc – or Mason–Stothers' Theorem – can be established in an elementary and beautiful way and also implies Fermat's Last Theorem for function fields. On the other hand, the solutions to "Fermat equation" in the case of degree 2 – also called Pythagorean triples – are related to the "additive" group law of the rational points of the circle.

14 One passes from the abc inequality to the ramification version of the Vojta Conjecture by considering a suitable covering of the projective line ramified at three points. This covering just happens to be the curve defined by the Fermat equation.

15 It is indeed a rewarding and pleasant exercise to translate the special case of the Vojta Conjecture referred to above into Schmidt's Subspace Theorem.

of inter-universal Teichmüller theory – is given by the following *seminal new insight* due to Mochizuki, see [GenEll] § 3[16]:

> *under the assumption of a certain (in fact nonexistent)* global multiplicative subspace $\mu_\ell < E$, *one may apply Faltings' theory of isogeny invariance of heights in the case of the ℓ-dilated quotient E/μ_ℓ to* obtain a certain bound on the height *(roughly) of the form* $\mathrm{ht}(E) \leqslant \log(\ell)/(\ell - 1) \leqslant 1$.

We refer to [EssLgc] Example 3.2.1 for details. The simulation of this (in fact nonexistent) global multiplicative subspace, which is implemented, in essence, by working with the collection of local multiple subspaces (who do exist), is at the core of inter-universal Teichmüller and is what leads to the categorification process (involving Frobenoids and Hodge theaters) in IUT geometry, see § Categorification of a Diophantine Problem. Simulating a global multiplicative subspace requires one, in effect, to "rearrange" the way in which the primes of a number field are distributed. This issue of "rearrangement of the distribution of primes" is precisely what led Mochizuki to introduce *mono-anabelian geometry*[17], see [EssLgc] Example 3.8.2 (iii) and (iv). Ultimately, mono-anabelian geometry also plays an important role in IUT via its use in a certain transport process, see § Mono-anabelian Transport, which allows one to derive the IUT version of the global height inequality (i.e., $\mathrm{ht}(E) \leqslant \log(\ell)/(\ell - 1) \leqslant 1$) discussed above, see § An Anabelian abc Inequality.

At a less essential level, the shift from the number field context of the Vojta Conjecture to the p–adic local one of (absolute p-adic) mono-anabelian geometry, as in § "Fukugen" - From Classical to Absolute Mono-anabelian, is prepared with a certain reformulation of the Vojta Conjecture. This reformulation allows one to shift the focus of one's attention – by working with certain local data such as a "compactly bounded set $\mathcal{K}$", which satisfies a certain $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-invariance property, for p a prime number – from the classical notion of "size", or "heights", of numbers in the sense of the real or complex numbers (which plays a fundamental role in the traditional approach to Diophantine geometry) to the p-adic valuations of a number field. We refer to [GenEll] § 2 for more details. Moreover, this reformulation has the effect of *recasting the Vojta Conjecture, which is both a centerpiece and a conceptual pinnacle in classical Diophantine geometry, as a seminal frontier of different nature.*

It may be of interest to note at this point that the creative process of Mochizuki is quite similar to *Grothendieck's principle of the "rising sea"*, see below and also the quote of Deligne in the introduction. Inter-universal Teichmüller geometry, the culmination of a 20-year journey initiated by Mochizuki when he was a PhD student in Princeton with Faltings in 1991, crystallizes multiple facets of his already innovative previous work in arithmetic geometry, see Fig. 11, and may be regarded as the completion of a program that imparts new significance to the three main components – multiplicative subspaces, p-adicization of Vojta, and properties of general elliptic curves – of Mochizuki's 2009 paper [GenEll].

> *"The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it... yet it finally surrounds the resistant substance."*
>
> – A. Grothendieck in [R&S] § 18.2.6.4 (translation by McLarty [McL07]).

[16] Another component of [GenEll] involves establishing various arithmetic properties for "general" elliptic curves in the moduli space $\mathcal{M}_{1,1}$ of elliptic curves, some of which underlie the IUT notion of initial Θ-data. Such initial Θ-data must be *explicitly constructed* in order to apply the IUT algorithm of § Log-theta wandering - Algorithms to relate distinct Frobenius-like objects.

[17] We also refer to Fig. 10 for a more classical analogy between Diophantine and "anabelian" geometry.

Anabelian Reconstructions - Schemes and Monoids. Anabelian geometry deals with the issue of reconstructing – or *"fukugen"* in Japanese – a space X from its group of étale paths, or the étale fundamental group Π_X of X, an object that functorially encodes certain symmetries arising from (the étale coverings of) X. The theory of the étale fundamental group, which was developed at the beginning of Grothendieck's "Séminaire de Géométrie Algébrique" ([SGA1], 1971), naturally induces *an intermingling between the discreteness of number theory (via the absolute Galois group* $\mathrm{Gal}(\bar{k}/k)$*) and the continuity of geometry (via* $\Delta_X \simeq \widehat{\pi}_1^{\mathrm{top}}(X)$ *and the topological paths on* X*)* in a sequence of fundamental groups, and indeed is (chronologically) *the first main example of Grothendieck's resolution of the tension between the discrete and the continuous realms*:

$$X \times \operatorname{Spec} \bar{k} \dashrightarrow X \longrightarrow \operatorname{Spec} k \qquad \underset{\exists \text{anabelian}}{\overset{\text{functorially}}{\xrightarrow{\ \pi_1(-)\ } \atop \xleftarrow{\ \ ?\ \ }}} \qquad 1 \longrightarrow \underset{\Delta_X}{\pi_1^{\mathrm{et}}(X \times \bar{k})} \longrightarrow \underset{\Pi_X}{\pi_1^{\mathrm{et}}(X)} \longrightarrow \underset{\mathrm{Gal}(\bar{k}/k)}{\pi_1^{\mathrm{et}}(k)} \longrightarrow 1.$$

Here $X \times \operatorname{Spec} \bar{k}$ can be be thought of as the space obtained by regarding the system of polynomial equations with coefficients in k that define the space X as a system of polynomial equations with coefficients in an algebraic closure $\bar{k}$ of k. The goal of anabelian geometry is to obtain a reverse process to this functorial construction. Figure 9 below describes the *classical bi-anabelian formulation* of this process and some of its variants.

Fig. 9. ***Anabelian Geometry reconstruction:*** *Can any morphism between the étale fundamental groups* Π_X *and* Π_Y *be reconstructed, essentially uniquely, from a morphism between* X *and* Y*?*
The mono-anabelian *version involves a single object* X*, while the* absolute *version imposes the constraint of forgetting the relation between* X *and* k*.*

For X *and* Y *varieties over* k*, is the natural map*

$$\mathrm{Isom}_k(X, Y) \to \mathrm{Isom}_{\mathrm{Gal}(\bar{k}/k)}(\Pi_X, \Pi_Y)_{\Delta_Y}$$

an isomorphism?

In the case where X is of dimension 0, the anabelian reconstruction of X corresponds to the reconstruction of a number field from its absolute Galois group and is known as the Neukirch–Uchida Theorem (~1977). In the case where X is of dimension 1, that is to say, when X is a curve, Grothendieck conjectured that this anabelian property holds for a certain type of curves, namely, the *hyperbolic* ones; this conjecture was solved in several steps by the Japanese school (Nakamura, Tamagawa, Mochizuki, 1990-1995).

In Diophantine geometry and the abelian, i.e., commutative, realm, one cannot help but notice the analogy between the above bi-anabelian formulation and Faltings' Isogeny Theorem (i.e., the Tate Conjecture) of Fig. 10 below.

Fig. 10. ***Faltings' Isogeny Theorem*** *is a fundamental step in the proof of the Mordell Conjecture and can also be understood as an "anabelian" result for abelian varieties – where the fundamental groups are replaced by their* abelian *counterparts, i.e.,* ℓ*-adic Tate modules* T_ℓ*.*

For A *and* B *abelian varieties over a number field* k*, the natural map*

$$\mathrm{Hom}_k(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathrm{Gal}(\bar{k}/k)}(T_\ell(A), T_\ell(B))$$

is an isomorphism.

The étale fundamental group framework of [SGA1] is developed in the context of Grothendieck's arithmetic geometry of rings/schemes. Inter-universal Teichmüller theory, on the other hand, involves an arithmetic geometry that goes beyond Grothendieck's arithmetic geometry of ring and scheme structures. In contrast to Grothendieck's original anabelian context (as in [Gro97]), which is *relative* and *bi*-anabelian, the IUT context relies on a general and thoroughly absolute approach, namely, *absolute mono-anabelian geometry*[18], where one is concerned with reconstruction techniques without any reference to an alternative scheme for the sake of comparison[19] or to a fixed base scheme.

[18] Despite the many papers published in this field since its initiation in 2003, this aspect of anabelian geometry and its very distinctive flavor have been, over the last 20 years, largely ignored outside Japan.

[19] One can also think in terms of the *functor of points approach*, which describes points of a space X in terms of arrows $\bullet \to X$.

※ *We refer the reader again to the general overview of anabelian principles in the classical [NTM98] and to the report [Hos22] on the latest advances of the field.*

§ "Fukugen" - From Classical to Absolute Mono-anabelian. The approach of Mochizuki's early work in anabelian geometry, as in [pGC], builds on the pioneering group-theoretic insight and techniques of the earlier anabelian approaches of Nakamura and Tamagawa, but introduces an important shift of perspective: exploiting Faltings' p-adic Hodge theory leads to a decisive breakthrough already in the classical non-mono-anabelian and non-absolute context. This approach provides a *canonical container* $\mathbb{P}(D_X)$ *for the reconstruction of spaces*, namely, the projective space associated to the vector space D_X of global differentials on the curve X, that allows one to construct rational points *à la* Tamagawa, in this case as limits of p-adically convergent sequences of points that appear in covers of the curve X, see [Fal98] for an overview.

This shift of perspective led, in effect, to a sort of *regeometrization of anabelian geometry*, from a discipline that relied primarily on classical arithmetic techniques over global number fields and function fields to a discipline that utilizes *techniques of p-adic geometry* such as line bundles and Chern classes. This more geometric approach has since become the standard anabelian framework and, moreover, may be regarded as a further step toward the sort of theory of heights in the style of Faltings, which plays an important role in § Inter-universal Teichmüller Geometry.

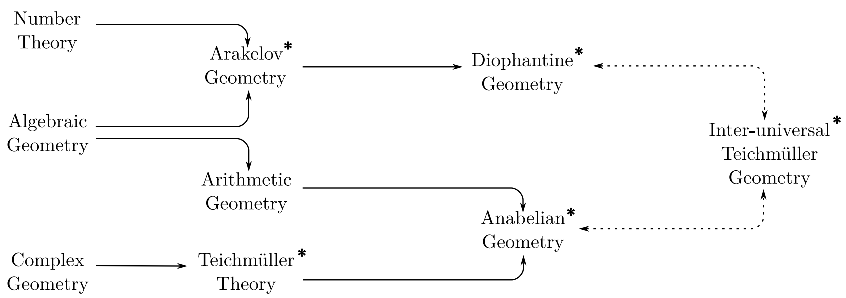


Fig. 11. ***Anabelian Geometry – a Contemporary Panorama.*** *Inter-universal Teichmüller geometry appears as a bridge between Diophantine geometry and anabelian geometry and as a new abutment of numerous tributaries originating in diverse fields of number theory and geometry; each* $*$ *indicates a major contribution by Mochizuki, often based on p-adic Hodge-theoretic considerations.*

Throughout anabelian geometry – i.e., including classical non-absolute, non-mono-anabelian versions of anabelian geometry, as well as more recent absolute, mono-anabelian developments – a fundamental role is played by the decomposition and inertia subgroups of the fundamental group, which are often denoted D_x and $I_x < \Pi_X$, respectively. In Mochizuki's absolute anabelian geometry, these decomposition and inertia subgroups of the fundamental group are reconstructed by means of the techniques of elliptic and Belyi cuspidalization, which may be thought of as "hidden symmetries" that relate a curve to covers of the curve with "missing" points.

For a survey that includes the most recent (at the time of writing of the present manuscript) progress in anabelian geometry, we refer to [Hos22]. For higher-dimensional anabelian results, see § 5 ibid. and also Note 21.

§ "There exists a group-theoretic algorithm...". While absolute mono-anabelian geometry concerns the reconstruction of a space X from a *single object* Π_X, it requires one to restrict one's attention to procedures and properties that are *common to all objects* considered[20]. Mono-anabelian geometry involves statements of the form:

> *"There exists a functorial group-theoretic* algorithm *to reconstruct* X *from* Π_X*", or "*$\Pi_X \rightsquigarrow X$*".*

[20] Regarding mono-anabelian geometry, Tamagawa speaks of a theory that deals with "one for all, all for one" properties and proposes the terminology of "omni" or "pan" anabelian geometry.

In this terminology, one works in an "absolute", as opposed to relative $\Pi_X = \pi_1^{\text{et}}(X) \to \text{Gal}(\bar{k}/k)$ context, in the sense that (the ring/scheme structure of) X *is detached from* (the ring/scheme structure of) its base field k.

In order to convey a sense of the flavor of a mono-anabelian (and absolute) reconstruction procedure, let us present, in the case of a hyperbolic curve X over a p-adic local base field k, a rough overview of the *mono-anabelian reconstruction of the base field k (via the function field K_X)*:

- (i) *Inertia and decomposition groups.* Belyi cuspidalization allows one to reconstruct, first, the decomposition group $D_x < \pi_1^{\text{et}}(U)$ associated to a (closed) point x of X, then the inertia group $I_x = D_x \cap \Delta_U < \pi_1^{\text{et}}(U)$ associated to x, where U is the complement in $X \smallsetminus \{x\}$ of a finite set of closed points.
- (ii) *Synchronization of geometric cyclotomes.* One reconstructs the canonical isomorphisms between geometric cyclotomes $I_x \simeq I_y$ for $x, y \in X \smallsetminus U$ that globally synchronize the (local) loops around various closed points.
- (iii) *The multiplicative groups $K_X^\times$ and $k^\times$.* One reconstructs the image of the Kummer map $\kappa\colon \Gamma(U, \mathcal{O}_U^\times) \hookrightarrow H^1(\Pi_U, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_X))$, where U is allowed to vary among the non-empty open subschemes of X. This yields a reconstruction of the multiplicative monoids $(k^\times, \boxtimes)$ and $(K_X^\times, \boxtimes)$, where $k^\times < K_X^\times$. Here $H^1(\Pi_U, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_X))$ plays the role of a common "container", and $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_X)$ is a synchronized cyclotome.
- (iv) *The field k.* One reconstructs the additive structure on the multiplicative monoid $(k^\times, \boxtimes)$ by considering the various quotients of the multiplicative monoid $(K_X^\times, \boxtimes)$ corresponding to valuations of K_X. This yields a reconstruction of the field $(k = k^\times \cup \{0\}, \boxtimes, \boxplus)$.

We refer to [AbsTopIII] § 1 for more details. One notes, in particular, the 2-step reconstruction of the $\boxtimes/\boxplus$-monoid structures *in a compatible* way, so that the $(\boxtimes, \boxplus)$-field structure of the base field k of the curve X is recovered. For a completely different situation in the zero-dimensional case, i.e., of a field that is not regarded as the base field of a hyperbolic curve, see § Mono-anabelian Transport below.

While most anabelian theorems for curves admit corresponding p-adic absolute anabelian versions (which are much stronger), it is important to note that such adaptations require a deep expertise in anabelian geometry[21].

Mochizuki's Belyi cuspidalization techniques lead in particular to *a new anabelian class of geometric objects*, namely, the curves of strictly Belyi type, see [Hos22] § 6. The algorithmic reformulations of absolute mono-anabelian geometry allow reconstructions that are *independent of a given fixed ring structure*, a feature that is exploited throughout inter-universal geometry.

§ Mono-anabelian Transport. Despite being the most natural context for anabelian geometry, the zero-dimensional case of a p-adic local field can already be quite subtle, since *the isomorphism class of such a field k is not necessarily determined by the isomorphism class of its absolute Galois group* $\text{Gal}(\bar{k}/k)$.

One fundamental point concerns the issue of giving a compatible reconstruction of the two underlying $\boxtimes$- and $\boxplus$-monoid structures of k. Indeed, while

> *there exists a functorial group-theoretic algorithm to reconstruct from the group* $G = \text{Gal}(\bar{k}/k)$ *the additive* $(\bar{k}_+(G), \boxplus)$ *and the multiplicative* $(\mathcal{O}_{\bar{k}}^\times(G), \boxtimes)$ $\text{Gal}(\bar{k}/k)$*-monoids*

– where $\bar{k}_+(G)$ denotes the underlying *additive* monoid of the field $\bar{k}$, and $\mathcal{O}_{\bar{k}}^\times(G) < \bar{k}_\times$ the multiplicative monoid of units of the ring of integers, both of which are reconstructed from G – it is important to note that *these two monoids cannot be reconstructed compatibly in such a way as to yield a reconstruction of the ring or field structure of $\bar{k}$*. A similar conclusion also holds for k.

[21] One example may be seen in higher-dimensional anabelian geometry, namely, in the case of results concerning the existence of an anabelian open basis — originally conjectured by Grothendieck — which may be naturally derived from techniques of the Japanese anabelian school via *p-adic absolute mono-anabelian* methods, see [Hos20] Cor. 3.4 & Rem. 3.4.1; see [SS16] for a result that holds only in the *relative* situation over number fields.

As a result, this zero-dimensional anabelian context goes beyond the classical ring-scheme and Galois theories and leads to a certain geometry of monoids. In this geometry, the two ⊠ and ⊞-monoids structures are reconciliated by allowing additional indeterminacies such as (Ind1) and (Ind2). Some new types of *étale-like objects*, e.g., the Galois group $\mathrm{Gal}(\bar{k}/k)$, and some *Frobenius-like objects*, the ⊠ and ⊞-monoids, naturally appear. A new kind of *mono-anabelian transport* process[22] defines isomorphism classes of these more rigid objects, see Fig. 12 for an example.

$$
\begin{array}{ccccc}
H^1(G_k, \boldsymbol{\mu}_{\bar{k}}^{\widehat{\mathbb{Z}}}(G_k)) & \xrightarrow{\sim *} & H^1(G_k, \boldsymbol{\mu}_{\bar{k}}^{\widehat{\mathbb{Z}}}(G_k)) & & \textit{étale-like} \\
\uparrow \kappa & \circlearrowleft \widehat{\mathbb{Z}}^\times & \downarrow \kappa^{-1} & & \\
\mathcal{O}_k^\times & & \mathcal{O}_k^\times & & \textit{Frobenius-like}
\end{array}
$$

Fig. 12. ***Mono-anabelian transport for multiplicative monoids of sub-p-adic fields*** *between Frobenius-like objects via isomorphisms between étale-like objects. The isomorphism obtained between Frobenius-like objects is subject to an $* = Aut(G_k)$-indeterminacy (Ind1) as well as a $\widehat{\mathbb{Z}}^\times$-indeterminacy (Ind2). Notation κ denotes the Kummer morphism.*

In inter-universal Teichmüller geometry, mono-anabelian transport is a seminal process that uses étale containers to share Frobenius-like data across a certain non-schematic Θ-link, see § Log-theta wandering - Algorithms to relate distinct Frobenius-like objects. In this context, the (Ind1) and (Ind2) indeterminacies, together with an additional (Ind3) indeterminacy that is discussed in loc. cit., can be regarded as the *mild deformations of the field structure* necessary in order to render compatible the (a priori incompatible) ⊠ and ⊞-monoid structures on opposite sides of the Θ-link.

This construction bears some resemblance to the analytic transport that appears in classical Grothendieck-Teichmüller theory between *two isomorphic but distinct* étale neighborhoods in the projective line, see Fig. 4. In this case, the analytic link requires no-indeterminacy but produces new arithmetic data, see also § Anabelian progress in Grothendieck-Teichmüller theory for a reverse application of anabelian geometry to Grothendieck-Teichmüller theory.

On the path to IUT geometry, this shift to a mono-absolute framework and indeterminacies is a key milestone in establishing a flexible geometry of monoids that allows one to deform, then estimate the discrepancy between the additive and multiplicative structures on opposite sides of the deformation (i.e., the Θ-link), which yields the inequality of the abc Conjecture.

※ *As we discuss later, one essential aspect of IUT involves the p-adic geometry of elliptic curves E_v over p-adic local fields $K_v/\mathbb{Q}_p$ and a certain p-adic theta function. The corresponding fundamental group is in this case given by* André's tempered fundamental group*, and the anabelian reconstructions involve* combinatorial anabelian geometry*, see also § Anabelian progress in Grothendieck-Teichmüller theory. These tempered reconstructions are indeed compatible with the profinite ones described above – we refer to [Lep10] § 1 for an introduction to Mochizuki's functorial results on this topic.*

[22] This elementary example indeed already involves the full techniques of poly-isomorphisms and synchronization of cyclotomes later used in the IUT § Log-theta wandering - Algorithms to relate distinct Frobenius-like objects. We refer to the introductory [Hos21] that was presented at the conference "Fundamental groups in arithmetic geometry" at the Institut Henri Poincaré in Paris in 2016 for further details and comments.

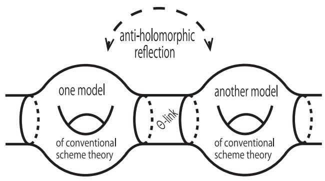
Inter-universal Teichmüller Geometry

While Grothendieck's *Éléments de géométrie algébrique* rewrote the foundations of algebraic geometry using *general ring theory* and *formal thickenings* (the deformation of scheme structures within ring/scheme theory), Mochizuki's inter-universal Teichmüller theory[23] studies *deformations*[24] *of the ring structure* by "untangling" the two ⊠/⊞-dimensions of a ring, *using the theory of monoids*, and then estimating the degree to which the ring structure may be reconstructed by "juggling" the ⊠/⊞-structures. Mono-anabelian geometry plays a central role in making this "juggling", which underlies the reconstruction of the ring structure, possible. In accordance with Grothendieck's philosophy, as discussed in § Grothendieck's Mathematics Philosophy, it is based on *the discovery of new category-theoretic structures and the creation of new language*, in a way that pushes arithmetic geometry beyond the algebraic geometry of rings and of Galois groups. By doing so, it finalizes the inclusion, initiated by Vojta, of Diophantine discreteness into the already existing discrete-continuous realm of anabelian geometry.

Fig. 13. ***Inter-universal Teichmüller theory****. A triangle between language, formalism and geometric pictures to resolve the tension between continuity and discreteness. The following excerpts are from the original IUT papers [IUTchI-IV]:*

"one may think of the fullness condition of multiradiality as the condition that there exist a sort of parallel transport isomorphism between two collections of radial data [i.e., corresponding to two "fibers"] that lifts a given isomorphism between collections of underlying coric data [i.e., corresponding to a path between the points over which the two fibers lie]."

$$({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_\alpha \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha;$$
$$({}^{n,m}\mathbb{M}^{\circledast}_{\mathfrak{mod}})_\alpha \xrightarrow{\sim} \mathbb{M}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha;$$
$$({}^{n,m}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_\alpha \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha;$$
$$({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_\alpha \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha;$$
$$({}^{n,m}\mathcal{F}^{\circledast}_{\mathrm{mod}})_\alpha \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha;$$
$${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}};$$



Inter-universal Teichmüller theory is not an incremental enhancement of Grothendieckian geometry, but rather a *fundamental conceptual advance based on new paradigms* – see also Note 33 for the étale fundamental group context discussed in § Anabelian Reconstructions - Schemes and Monoids and, even more generally, the discussion in § Universes, Species, and Logical Structure – that provides new insight into (1) the interface between anabelian and Diophantine geometry, and (2) the relationship between ring structures and underlying monoid structures in arithmetic geometry, with (3) new techniques for defining a new research frontier, see § New Anabelian & Diophantine Frontiers... These three aspects can be seen as part of *a second regeometrization of arithmetic geometry* – see § Introduction of [MFO21] and the report [Moc23] – this time *via* anabelian geometry.

The category-theoretic approach of inter-universal Teichmüller geometry results in a rich and evocative language for guiding mathematical thinking[25]: objects exists in *étale-like* and *Frobenius-like* flavors – depending on whether one regards an object as anabelianly reconstructed, e.g., from Galois groups, or, alternatively, as an object that is only defined relative to a particular ring or monoid structure. Moreover, both étale-like and Frobenius-like objects come in two variants, namely, *holomorphic* and *mono-analytic*, depending on whether they involve two or only one of the two monoidal structures of a ring, see the discussion of [Alien] § 2.7 (vii) and § 3.6 (iv).

[23] Mochizuki's IUT was first officially presented to the international community during the 2010 "Development of Galois-Teichmüller Theory and Anabelian Geometry" conference [Moc10] – see also the 2004 "Arithmetic Geometry" Tokyo international conference [Moc04] for a discussion of this work at a very preliminary stage. The initial versions of the IUT manuscripts appeared in August 2012 on the official public RIMS preprints server [RIMS1756, 1757, 1758, 1759] and were later updated on its author's web page following comments of the community and the referees (which resulted in 160 additional pages and 10 revisions of the submitted manuscript).

[24] We refer to the discussion surrounding (RdVar) in [EssLgc] § 3.1 for the analogy between the classical deformations of scheme theory and the deformations that appear in inter-universal Teichmüller theory.

[25] Consider, for instance, the following typical IUT statement: "The (Ind2) indeterminacy arises from the passage from mono-analytic Frobenius prime-strips to mono-analytic étale prime-stripes." We refer to Marquis' [Mar22] on the cognitive value of a mathematical style (illustrated in loc. cit. in the case of Bourbaki's structuralism).

CATEGORIFICATION OF A DIOPHANTINE PROBLEM. The natural geometric context of IUT is that of elliptic curves that belong to certain subsets of (the set of rational points of) the associated moduli space[26] $\mathcal{M}_{1,1}$, i.e., over a global field K, where the goal, in order to establish a Vojta-like inequality, is to construct *a certain global multiplicative subspace* $\mu_\ell < E$, see § From Vojta to Mochizuki - Geometrization & Arithmetization.

To achieve this goal, one must overcome two issues, which in fact constitute essential guiding principles in IUT theory:

(i) While such a subspace exists locally – i.e., at certain places v, for elliptic curves $E_v \in \mathcal{M}_{1,1}(K_v)$ over local fields K_v – such a subspace typically *does not exist globally*, i.e., for $E \in \mathcal{M}_{1,1}(K)$.

(ii) The quotient morphism $E \to E/\mu_\ell$ – that gives rise to the height inequality, see ibid. – amounts, at the level of moduli spaces, to a sort of Frobenius morphism $(-)^\ell$ *that is not a ring homomorphism* – e.g., $(a+b)^\ell \neq a^\ell + b^\ell$.

In order to overcome these obstructions, Mochizuki proposes (1) *a categorification of the original situation* by constructing Hodge theaters[27] $\mathcal{HT}^\bullet$ that *simulate the existence of such a* global *multiplicative subspace* $\mu_\ell < E$ – see [EssLgc] Example 3.2.1 – and (2) an *arithmetic geometry that does not depend on ring structures.*

Such a categorification of the arithmetic-geometric framework can be seen as *a second example of Grothendieck's principle of resolution of the tension between the discreteness of number theory and the continuity of geometry*, which this time, leads to the inclusion of the realm of Diophantine estimates into the realm of anabelian geometry – for the first example, we refer to § Anabelian Reconstructions - Schemes and Monoids.

※ *Before going further, we recall that over a p-adic local field, an elliptic curve E_v with bad reduction (1) may be identified with the Tate curve $\mathbb{G}_m/q_{E_v}^{\mathbb{Z}}$ defined by its q_{E_v}-parameter and (2) admits a canonical function, called the Θ-function, that is determined by certain symmetry properties (studied by Mumford and Tate). Indeed, the main results of Mochizuki's [EtTh] establish some absolute mono-anabelian reconstruction algorithms for this p-adic Θ context*[28].

§ Hodge Theaters & Synchronization between Geometry and Arithmetic. Not only does IUT fully exploit the category-theoretic setting of Grothendieck's SGA1 theory of étale fundamental groups – at the interface of number and algebraic geometry theory in the most concrete way, see Fig. 14 and Note 30 – it also extends this geometry beyond Grothendieck's framework of rings, fields, and Galois groups by deforming the $\boxtimes/\boxplus$-monoid structures of a ring.

The *category-theoretic simulation of a global multiplicative subspace* $\mu_\ell < E$ is realized by means of $\Theta^{\pm\text{ell}}$NF-Hodge theaters. These new category-theoretic objects appear as the result of an elegant synchronization process involving:

(i) the *geometric symmetries* of (a certain covering of) a 1-pointed genus 1 curve X (i.e., a sort of hyperbolic version of the elliptic curve E), and

(ii) the *arithmetic symmetries* of the number field K which is the base field of X – see Fig. 14[29].

---

[26] The projective line minus three points $\mathbb{P}^1 \setminus \{0,1,\infty\}$, which was mentioned earlier in § "abc" - The Shadow of a Network of Conjectures, is a coarse version of this moduli space.

[27] Formally, a Hodge theater is composed of Frobenioids – that category-theoretically encode the geometry of line bundles – assembled into various kinds of prime-strips that gather local and local-to-global data related to étale-like objects and Frobenius-like objects. We refer to Fig. I1.2 of Chap. I of [IUTchI-IV] for examples of prime-strips.

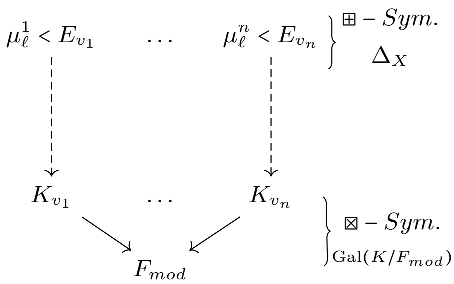
[28] ... and a remark similar to the one of § From Vojta to Mochizuki - Geometrization & Arithmetization on the recasting aspect of Mochizuki's work on Vojta in [GenEll] applies to the context-related significance of this work, i.e., it furnishes a new seminal perspective on classical mathematical objects.

[29] The $\boxtimes$-symmetries result from the need to work at the level of (the places v_i of) K, rather than the (perhaps somewhat more natural) field of moduli F_{mod}. The synchronization is realized by working with certain coverings that support the $\boxplus$-symmetries. We refer to [EssLgc] Example 3.8.2 for a detailed presentation.

Such a Hodge theater $\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ is obtained by "gluing"[30] together a ΘNF-part $\mathcal{HT}^{\Theta\mathrm{NF}}$ – NF for "Number Field" – to a $\Theta^{\pm\mathrm{ell}}$-part $\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$ – where Θ refers to the Θ-function attached to the elliptic curve under consideration. The realm of $\mathcal{HT}^{\Theta\mathrm{NF}}$ is *Galois-theoretic/arithmetic and multiplicative*, while the realm of $\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$ is *geometric and additive*, we refer to [Alien] § 3.3 (v) for further details on these symmetries and the gluing construction.

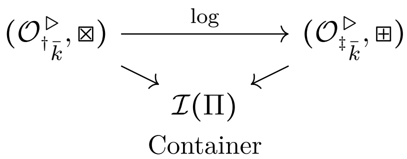
In this category-theoretic context, the "ℓ-dilation" of the ⊠-monoid structures[31], as in § From Vojta to Mochizuki - Geometrization & Arithmetization, is obtained via a Θ-link that operates at the level of Frobenius-like objects, see Fig. 12. This Θ-link is essentially defined, on the q_E-parameter of a Tate elliptic curve E, as the Frobenius morphism $q_E \mapsto q_E^{j^2}$ for j an integer – more precisely, a sequence of such $q_E \mapsto q_E^{j^2}$, where $j \in \{1, \ldots, (\ell-1)/2\}$. While the Θ-link is compatible with the ⊠-monoid structures and determines a gluing between two isomorphic copies of the same $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, the Θ-link is not compatible with the ⊞-monoid structures. We thus denote the two *distinct but isomorphic ring-structures* involved with *distinct labels* ${}^{\dagger}(-)$ and ${}^{\ddagger}(-)$ – or more generally with an (n, m) indexing – so that one remembers that this framework is not compatible with a *single fixed* field-ring structure.

Fig. 14. *Local description of the two types of symmetry that underlie the simulation of the global multiplicative subspace.*



In order to desynchronize-resynchronize the Θ*-link deformation of the multiplicative structure with respect to the (fixed) additive structures of the rings on either side of the* Θ*-link*, IUT relies on the use of a certain sequence of log-links on either side of the Θ-link. Each such log-link relates the ⊠-monoid structure in its source – denoted $(\mathcal{O}^{\triangleright}_{\bullet}; ⊠)$ – to the ⊞-monoid structure in its target – denoted $(\mathcal{O}^{\triangleright}_{\bullet}; ⊞)$ – by embedding both in a common log-shell container $\mathcal{I}(-)$, whose introduction becomes *unavoidable as soon as one attempts to deal with the* ⊠*- and the* ⊞*-monoid structures together simultaneously*, see Fig. 15. Here Π denotes a certain étale fundamental group that forms the input data in the absolute mono-anabelian reconstruction of the container objects in such a way that Π is detached from the structure morphism $\Pi \to \mathrm{Gal}(\bar{k}/k)$ (which, a priori, depends on the ring structures involved), so that *the ring structure can be freely deformed* during mono-anabelian reconstructions as in § Mono-anabelian Transport.

Fig. 15. *The log-link, the relevant* ⊞/⊠*-structures, and the log-shell container* $\mathcal{I}(\Pi) \subseteq \bar{k}(\Pi)$.



Finally, we should emphasize that, in contrast to Grothendieck's SGA1, which is essentially a theory of the fundamental group *attached to a single (geometric) base point*, the anabelian reconstruction algorithms of IUT geometry involve *multiple base points*[32], as a result of the variation of the ring structures involved. Indeed, this *non-existence of a unique base point* is a consequence of the following three components of IUT: (a) the internal structure of Hodge theaters, which involves the synchronizations of certain conjugates (see Note 30, above), (b) the Θ-link, which involves a dilation of the ring structure, and (c) the p-adic logarithm, which permutes, in a complicated fashion, the ⊠- and ⊞-monoid structures, see ibid. Example 3.8.3 (i-d), (i-e), and (vi).

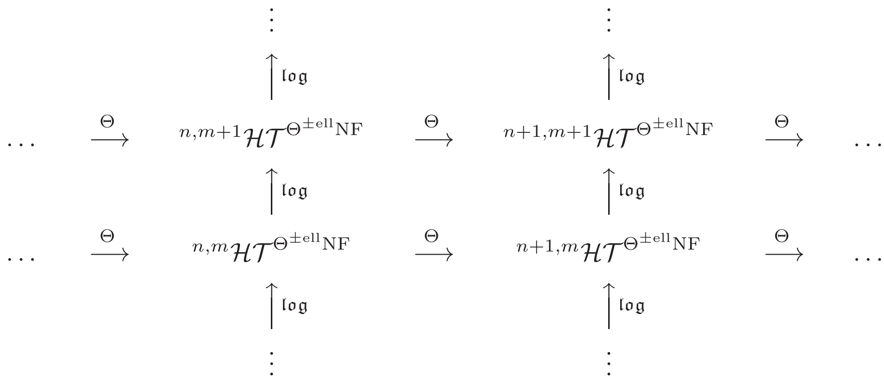
30 In particular, this construction makes possible the synchronization, for each valuation v, of the unavoidable G_v-conjugate indeterminacies that originate from the functoriality of the SGA1 constructions, see [EssLgc] Example 3.8.2 (iii) and 3.8.1.

31 This dilation of analytic structures can indeed be seen as the origin of the term "Teichmüller" in the terminology "Inter-universal Teichmüller Theory".

32 In Grothendieck-Teichmüller theory already, see Fig. 4, the arithmetic importance of dealing with multiple base points is usually ignored by non-anabelian arithmetic geometers (and thus mistakenly described as "a gadget" or "a trick"). In IUT theory, the use of multiple base points is an unavoidable requirement, as discussed in [EssLgc] Example 3.8.3.

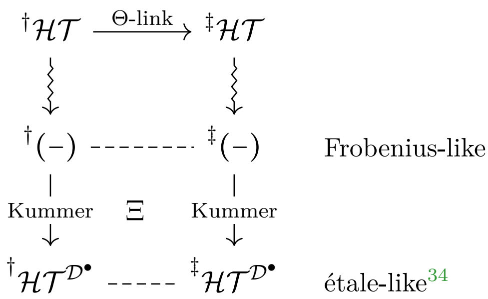
§ Log-theta wandering - Algorithms to relate distinct Frobenius-like objects. In this category-theoretic setting, the above constructions[33] are put together to form *the log-theta lattice of Hodge theaters* ${}^{(m,n)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, for $m, n \in \mathbb{Z}$, see Fig. 16. The construction of structures, via a certain multiradial algorithm, that are invariant with respect to (certain types of) *highly non-commutative wandering in the log-theta lattice* forms the main theorem of IUT, see Fig. 17 and below. The situation is qualitatively similar to the mono-anabelian transport situation of Fig. 12 and more generally of § Mono-anabelian Transport (in the case of wandering within a single column, we refer to Fig. 15): Hodge theaters admit an étale-like variant, linked to the original Hodge theater via Kummer morphisms, also up to certain indeterminacies.

Fig. 16. ***Log-theta lattice of Hodge Theaters.*** *In the shadow of this lattice, an additional sub-layer of étale-like containers "Kummer-connects" the Hodge theaters in each vertical column. The multiradial algorithm allows one to* navigate from this sub-layer to a top-layer *that consists of the Frobenius-like components of the lattice.*



The *Main Theorem of inter-universal Teichmüller theory* states the existence of a multiradial algorithm at the level of Hodge theaters, see Fig. 17 and Fig. 18, as follows:

Fig. 17. ***The Main Theorem of inter-universal Teichmüller theory (multiradial algorithm).*** *Given a certain arithmetic-geometric context for elliptic curves, there exists an algorithm* Ξ *for constructing a common container for the Frobenius-like data associated to distinct Hodge theaters, labeled* ${}^{\dagger}(-)$ *and* ${}^{\ddagger}(-)$*, that are related to one another by the non-schematic* Θ*-link.*

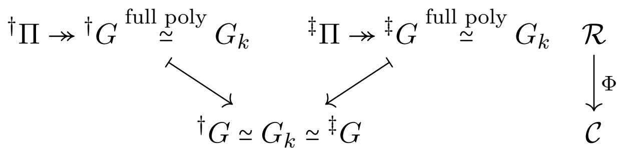


The arithmetic-geometric context for elliptic curves mentioned above is given by fixing the so-called *initial* Θ*-data*; the common container corresponds to the étale-like data mentioned above; the algorithm relies on the absolute mono-anabelian reconstructions discussed in § "There exists a group-theoretic algorithm...". We refer to [Alien] § 3.2 for additional examples of multiradiality. The algorithm of Fig. 17 is defined up to certain indeterminacies (Ind1), (Ind2), and (Ind3) that originate as follows: **(Ind1)** comes from comparing mono-analytic étale-like objects by mono-anabelian transport; **(Ind2)** comes from a mono-analytic étale-like vs. Frobenius-like comparison; finally, **(Ind3)** originates from the non-commutativity of the log-Kummer correspondence. Of these, *(Ind 3) is the most essential indeterminacy of IUT* and plays the most important role when the algorithm of the Main Theorem of IUT is applied to the log-theta lattice of Hodge theaters for *establishing the height inequality*, or Vojta-like inequalities, of the abc Conjecture – see the discussion of Θ-pilot/q-pilot objects in § An Anabelian abc Inequality.

[33] In this monoid-theoretic setting, one could ask what has become of the SGA1 "automorphism group of a fiber functor over a geometric base point" ring-theoretic setting. We refer to the discussion on "a single unified basepoint" and "single unified set-theoretic basepoint" of [EssLgc] Examples 3.8.3 and 3.8.4 – see also Example 3.8.1 ibid.

[34] This figure is adapted from Minamide's talk "Log-Theta Lattice: Symmetries and Indeterminacies" in [ExpHoriz1].

Fig. 18. ***A Multiradial Algorithm*** *allows one to recover objects $\Pi \twoheadrightarrow G$ that appear in the top row from objects constructed from the bottom row up to certain indeterminacies; ${}^{\dagger}(-)$ and ${}^{\ddagger}(-)$ denote isomorphic but distinct copies of isomorphic groups.*



In Mochizuki's own words, an essential inspiration for the notion of a multiradial algorithm is Grothendieck's definition of *the notion of a connection in the theory of crystals* – see [Alien] § 3.1 (iv)-(v) for details. Both notions may be regarded as a sort of *a descent property* – see [EssLgc] § 3.9.

§ Universes, Species, and Logical Structure. In the case of arithmetic geometry breakthroughs – such as, in the abelian cohomological context, the proof of Fermat's Last Theorem by Wiles, and indeed already the proof of the Weil Conjectures by Deligne, see Mclarty's discussion [McL10] and especially § 7 ibid. – the issue of identifying their underlying set-theoretic foundational framework – e.g., Peano, Zermelo-Fraenkel, or ZFC + the existence of Grothendieck universes – is of specific importance with respect to Grothendieck's philosophy of mathematics since *it is categories, not objects, that provide a seminal context for the virtuous practice of mathematics* – see also Note 36.

The foundational basis of IUT geometry can be approached at two distinct levels: an "external" one that deals with the proper and logical articulation of sequences of statements, and an "internal" one that ensures the correct interaction of objects, morphisms, categories, and functors that appear in the theory. As presented in detail and in multiple contexts in [EssLgc], the former "external one" boils down to a sequence of logical "OR" and "AND" relations[35] – we refer to *§ 3. The logical structure of inter-universal Teichmüller theory* ibid.

The "internal" level, which may also be described as category-theoretic, is related to Grothendieck's notion of "universe", which provides a foundational framework in set theory and amounts to fixing a ZFC model[36]. In the Main Theorem of IUT of Fig. 17, it must be noted that each application of the mono-anabelian reconstruction algorithms that appear involves *a potential change of universe* by successive enlargement (and thus *a priori* incompatible height or degree comparison, see the discussion below concerning species and mutations, as well as § An Anabelian abc Inequality). As examined in Chap. IV "Log-volume Computations and Set-theoretic Foundations" of [IUTchI-IV], "albeit from an extremely naive/non-expert point of view!" (dixit Mochizuki, ibid.)[37], this issue in IUT is taken care of *via* the notions of *species and mutations*[38] – such as, respectively, the $\mathcal{R}$ (or $\mathcal{C}$) and Φ in Fig. 18. Species and mutations are respective analogues of categories and functors, but *defined in a sound set-theoretical framework* to provide a foundational apparatus for the various mono-anabelian and inter-universal Teichmüller reconstruction algorithms, which can thus eventually be expressed without any assumption on *a fixed choice of universe* – we refer to the absolute anabelian and étale-Frobenius Examples 3.4-3.6 ibid. The following is a reformulation of Remarks 3.1.4 and 3.6.3 ibid.:

35 While a modelization of IUT via proof assistants, such as Lean or Coq, would thus be of very limited interest, it could reveal finer arithmetic-geometric structures in-between the discrete and continuous realms – we refer to the Coq proof of the Four Colors Theorem by Gonthier et al (2005) and the introduction of combinatorial hypermaps as presented in [Oli23].

36 In Grothendieck's mathematical practice and philosophical view, the notion of universe may be considered as a "conceptual gadget" for resolving Bourbaki's foundational category-theoretic problems, not as a virtuous ground for subsequent developments, see Marquis' contribution in [Chap23] and also [Krö06] § 6 – such a virtuous role is indeed explicitly attributed to *topoi.* By contrast to the category-theoretic prism, one notes that IUT is not a geometry of commutative diagrams – see Examples 3.6 (Syp2) and 3.10.2 in [EssLgc].

37 The reader will allow the author to take an even more naive position, whose goal is only to bring these considerations to the eyes of experts in the hope of a potential formal and rigorous treatment.

38 While *a priori* distinct, both Bourbaki's and Mochizuki's notions of species can be seen as illustrating the same issue of defining a set-theoretic foundation for the category-theoretic and functorial practice of mathematics, see [Krö06] § 6 and [Mar22].

> *Although the mono-anabelian reconstruction algorithms can* a priori *give rise to numerous* enlargements of universe – *and thus* a priori *numerous ZFC models – the foundation of IUT in terms of species and mutations ensures that the final IUT algorithm only relies on* their shared portion, i.e., on a certain "inter-universal" model.

Species and mutations further ensure that the non-scheme/ring-theoretic Θ-link and log-link are well-defined in a sense that is independent of a fixed choice of ZFC model.

With regard to heights, or arithmetic degrees, because of the non-ring-theoretic anabelian reconstruction procedures involved *at each local place* v, it should be noted that each local height $\deg_v$ takes values, strictly speaking, in a copy $\mathbb{R}_v$ of the real numbers that belongs to *a different ZFC model*. In this anabelian context, the notion of universe is thus a *substantive seminal* concept which is of a completely different nature from its auxiliary ("Helper") role in establishing abelian cohomological arithmetic results – see [McL10] regarding Wiles' FLT and Deligne's Weil Conjectures, see § An Anabelian abc Inequality regarding IUT.

Let us remark that if one follows Grothendieck's viewpoint, as discussed in Note 36, even further, it would be reasonable to expect for the *theory of topoi* – a theoretical framework that bridges category theory, logic, and geometry – to provide an even finer and more revealing framework for these changes of ZFC model and universe – we refer to [Ble22] and references therein for an accessible and introductory presentation of topoi from the perspective of logic and mathematical universes.

> *"[Mochizuki's IUT] work [...] puts anabelian geometry on a higher level/perspective and opens new unprecedented horizons in the research surrounding arithmetic fundamental groups and their connection to diophantine geometry [...] The fact (established in these papers) that Galois theory, and more precisely anabelian geometry, has a particular control on certain diophantine inequalities is an extraordinary fact which is unprecedented."*
> – M. Saïdi in [MR4225476]

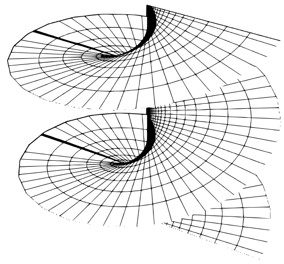
NEW ANABELIAN & DIOPHANTINE FRONTIERS.... In inter-universal Teichmüller theory, the abc inequality becomes an estimate on *how far the additive and multiplicative structures of a ring can be dismantled from one another.* Not only does the categorification approach of Mochizuki lead to an anabelian proof of the abc Conjecture and of Fermat's Last Theorem, it also provides some seminal "reverse" insight into classical anabelian arithmetic geometry, or "Galois-Teichmüller theory". The latter is indeed a natural context in which to test new insights of anabelian arithmetic geometry.

§ An Anabelian abc Inequality. The abc inequality – also the first proof of the Vojta Conjecture "with ramification" in the case of curves – results from establishing the "Vojta-Mochizuki inequality" (i.e., [GenEll] Theorem 2.1), see also § From Vojta to Mochizuki - Geometrization & Arithmetization. In IUT, this inequality follows from two inequalities between the arithmetic degrees – or "heights", or log-volumes – of *two regions in containers arising from the log-theta lattice of Hodge theaters* of Fig. 16, that are described as *a certain Θ-pilot object* Θ_E *and a certain q-pilot object* q_E:

$$\begin{cases} -|\deg q_E| \leqslant -|\deg \Theta_E| < +\infty \\ -|\deg \Theta_E| \leqslant a_\ell - b_\ell |\deg q_E| \text{ with } a_\ell,\ b_\ell \in \mathbb{R} \text{ and } b_\ell > 1 \end{cases} \text{and thus } \deg q_E \leqslant a_\ell/(b_\ell - 1), \quad \text{(HtIneq)}$$

where the first inequality[39] follows from *the Main Theorem of IUT* on the existence of a multiradial algorithm as in Fig. 17 and Fig. 18, and the second one follows from some direct computations in classical number theory, see Theorem 1.10 of Chap. IV of [IUTchI-IV]. It is interesting to observe, once again, that in order to apply this argument, it is necessary to construct some explicit initial Θ-data, as in Corollary 2.2 ibid., that are related to concrete arithmetic and geometric properties of elliptic curves. This sort of tight *cohesion of IUT between the category-theoretic, geometric, and computational realms* is a trademark of the soundest and most impactful mathematical theories.

[39] For the original statement, we refer to Corollary 3.12 of Chap. III of [IUTchI-IV], where the notation "log(−)" is used instead of "deg", as in the present discussion.



The Θ-*pilot and q-pilot objects may be thought of as regions* inside some intricate collection of systems of tensor products of log-shells that arise from *local portions*[40] of the various Hodge theaters involved; *log-shells serve as common containers for the ⊞/⊠-monoid structures.* There is a certain analogy between the *log-theta lattice wandering* that gives rise to the comparison of Θ-pilot and q-pilot arithmetic degrees, up to certain indeterminacies, and the classical theory of *analytic continuation*, which involves a collection of multiple neighborhoods taken in various layers of the Riemann surface defined by the multi-valued complex logarithm – see the figure on the right.

Each Hodge theater contains various *global realified Frobenioids*, each of which may be understood as a collection of local copies $\{\mathbb{R}_v\}_{v\in\mathbb{V}(K)}$of $\mathbb{R}$ at each of the valuations v of the number field K under consideration *that are related to one another* via a relationship that is analogous to the product formula in elementary algebraic number theory. In particular, such a global realified Frobenioid gives rise to *a global copy of* $\mathbb{R}$, whose elements may be thought of as *global heights* deg obtained as sums of various local heights $\deg_v$. One Frobenius-like copy of such a global realified Frobenioid plays a central role in the definition of the *non-ring-theoretic* Θ-link and hence in the proof of the main IUT inequality between the heights of the q-pilot and Θ-pilot objects as in Eq. (HtIneq). With regard to the foundational aspects of the situation, as discussed in § Universes, Species, and Logical Structure, we observe that:

> *Because of the non-ring-theoretic constructions involved, the various Hodge theaters* ${}^{(m,n)}\mathcal{HT}^{\Theta^{\pm ell}\mathrm{NF}}$*, for* $m,n\in\mathbb{Z}$*, in the log-theta lattice involve distinct and incompatible ring structures — hence, in particular,* distinct and incompatible *copies of* $\mathbb{R}$*, i.e., that arise from distinct and incompatible global realified Frobenioids — which exist* a priori *in distinct universes (hence distinct ZFC models). It is only after taking into account certain indeterminacies that the multiradial algorithm of Fig. 17 is able to produce structures,* in a common universe (hence a common ZFC model)*, that are compatible with these distinct and incompatible ring structures and hence allow one to compare in a meaningful way — and, in particular,* to verify the height inequality *between — the q-pilot and* Θ*-pilot objects*[41] *as in Eq.* (HtIneq)*.*

Regarding the arithmetic significance of considering *isomorphic but non-identical objects* in anabelian geometry, we refer to the classical example of Grothendieck-Teichmüller theory in Fig. 4. In the case of IUT geometry, the non-identification of isomorphic objects allows one to distinguish and then compare various Frobenius-like objects[42]. The seminal role of such non-identifications may be understood in more detail by considering an alternative "RCS-IUT theory" – i.e., a theory where isomorphic objects are identified, and which may be proven to be *logically unrelated to the original IUT theory* – where the identification of isomorphic objects indeed produces an essential contradiction. Here, "RCS" stands for the "Redundant Copies School" of thought, see [EssLgc].

The multiradial construction of the Θ-pilot object results from a certain wandering within the log-theta lattice that allows one to reconcile the additive structures associated, via the respective

[40] We remind the reader that these local portions are independent of one another and not constrained to any fixed relationship with a global number field or indeed any other fixed ring structure.

[41] From the point of view of the maturation of mathematical ideas, it is interesting to remember (1) that an animated debate, led by Weierstrass, caused a stir in the mathematical community of the mid-19th century on the question of *"how isomorphic (non-labeled) neighborhoods in a Riemann surface could produce any new geometric information?"* (the "geometric fantasies", ~1864), and (2) that the dispute between Leibniz and Bernoulli on the logarithm of non-positive real numbers was resolved by the introduction of the *argument-indeterminacy* (~1751), see the discussion in [EssLgc] § 1.5. We also refer to [Moc18] (C17) for a discussion of what happens if one ignores the (m,n) labeling issue and does not apply the whole IUT algorithm.

[42] We also refer to [EssLgc] Example 2.4.8, which explains how this issue can be illustrated via the elementary notions of rings and monoids.

(distinct and incompatible) ring structures involved, to the dilated and undilated multiplicative monoid structures; *the three IUT indeterminacies (Ind1), (Ind2), and (Ind3) are just flexible enough for this to happen.*

§ An Extended Fermat's Last Theorem. As recalled in § "abc" - The Shadow of a Network of Conjectures, while it is natural to expect for a theory establishing abc to lead to a proof of Fermat's Last Theorem (FLT), in order to obtain such an application to FLT, it is necessary to establish *an "effective" version of the abc Conjecture, i.e., where the constant* K_ε *is explicitly given*, see [GT02] – a condition that requires an essential enhancement of the non-effective anabelian-Diophantine result of the original IUT.

Fig. 19. ***Fermat's Last Theorem.*** *For* $n > 2$ *the equation*

$$x^n + y^n = z^n$$

has no positive integer solution.

The main obstruction to establishing such an effective version lies in the initial use of a compactly bounded subset $\mathcal{K} < \mathbb{P}^1 \setminus \{0, 1, \infty\}$ – see the final portion of § From Vojta to Mochizuki - Geometrization & Arithmetization – that supports the prime 2, and which, in the original version of IUT, is later eliminated by applying *the (non-effective!) theory of noncritical Belyi maps.* In [ExpEst], the original version of IUT is refined by applying a construction that originates in [Por20], which constructs an étale theta function – and establishes associated properties of the corresponding monoids – via evaluation at 6-*torsion points* instead of at 2-torsion points as in the original Mumford construction and thus eliminates the requirement of avoiding the prime 2 via the use of the compactly bounded set $\mathcal{K}$.

The result is an *effective* version of abc, which in turns implies an anabelian proof of Fermat's Last Theorem, see [ExpEst], Theorem B and Corollary C:

$$|abc| \leqslant 2^4 \cdot \mathbf{exp(1.7 \cdot 10^{30} \cdot \varepsilon^{-166/81})} \cdot \mathrm{rad}(abc)^{3(1+\varepsilon)}, \text{ and } \textit{Fermat has no solution for } p > \mathbf{1.615 \cdot 10^{14}}$$

This lower bound – combined with a numerical result of Coppersmith (1990) and some new cyclotomic estimates of Mihailescu and Rassias (2022) – is indeed lowered to $p > \mathbf{257}$, hence yields, when combined further with a classical result of Vandiver (1929), a new proof of FLT, as well as a certain generalized version of FLT – see Cor. 5.9 ibid.

§ Anabelian progress in Grothendieck-Teichmüller theory. Let us mention another noteworthy example in Mochizuki's work of Grothendieck's principle of the categorification of arithmetic-geometric contexts – as discussed in § Grothendieck's Mathematics Philosophy, as well as in § Categorification of a Diophantine Problem – namely, *combinatorial anabelian geometry*[43], a theory developed by Mochizuki jointly with Hoshi (and later with Minamide and Tsujimura). We refer to [Hos22] § 7-8 and the references therein for a survey.

Fig. 20. ***GT theory*** *- an arithmetic-combinatoric-geometric triangle.*

$$\begin{array}{ccc} \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & \hookrightarrow & \mathrm{Aut}[\pi_1(\mathcal{M}_{0,[m]})] \\ & \searrow & \uparrow \\ & & \widehat{GT} \end{array}$$

Following Grothendieck's "Esquisse d'un programme" [Esq], Grothendieck-Teichmüller theory (GT) concerns the search for a combinatorial description of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ in terms of a group $\widehat{GT}$ that arises from the (geometric) étale fundamental group of the moduli stack $\mathcal{M}_{0,[m]}$ of genus 0 curves with m marked points – see Fig. 20, where the horizontal map is arithmetic, the vertical map is geometric, and the diagonal map is Galois-combinatoric. The leading question here is to determine *"how close are these three maps to being isomorphisms?"*

While Galois-Teichmüller theory and Grothendieck-Teichmüller theory have typically, in the past, provided some *input for anabelian geometry*, Mochizuki's school recently obtained two unexpected applications in the reverse direction, i.e., of *combinatorial anabelian geometry* to GT theory: (1) a result to the effect that $\widehat{GT}$ is larger than was previously expected – see [CbGT]; and, most essentially,

43 To avoid any confusion or potential misunderstanding, let us state clearly that *combinatorial anabelian geometry is a theory in its own right which does not involve any logical dependence on inter-universal Teichmüller theory.*

(2) *a combinatorial description* $\bar{\mathbb{Q}}_{BGT}$ *of the algebraic closure* $\bar{\mathbb{Q}}$ *of the field of rational numbers and hence of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *itself*, see [CbGal].

"En guise de conclusion"

Inter-universal Teichmüller theory opens a new chapter in arithmetic geometry in a way that faithfully follows and incorporates the most fundamental aspects of the philosophy of Alexander Grothendieck concerning the practice of mathematics. Given the multi-layered coherency of its geometric, category-theoretic, and explicit constructions – as well as how deeply rooted it is in the most essential techniques and progress of classical algebraic geometry and number theory – one may say that IUT comes across as a stimulating and virtuous theory for the mind of the arithmetic geometer.

IUT already opens new horizons *"internally"* – for example, in the form of refinements of the original version of IUT involving elliptic curves over number fields, as well as *degenerations of elliptic curves over number fields*, that are expected to lead to new number-theoretic applications that were not within the range of applicability of the original version of IUT – but also *more broadly*, as discussed in § Anabelian progress in Grothendieck-Teichmüller theory, with ramifications for other related topics in arithmetic and homotopic Galois theory. As was already reported in [Hos21b], a "Galois-orbit version of inter-universal Teichmüller theory" that is currently under development for *hyperbolic curves of arbitrary genus* is expected to yield important progress on the *local-to-global Grothendieck Section Conjecture* – for an introduction to this conjecture, see [Saï12] and [Hos14]. We further refer to [Moc23] § 4 for a report on work in progress and connections with other theories such as algebraic geometry (via "resolution of nonsingularities" in the sense of Tamagawa, see also for example [Lep13]) or analytic number theory.

One can thus only expect Mochizuki's inter-universal Teichmüller theory, as well as the general philosophy surrounding this theory, to act as one of the long-term beacons in the ongoing general harmonization process between arithmetic, on the one hand, and homotopic Galois theory, on the other.

※ *For additional introductory notes on inter-universal Teichmüller theory, we refer to the* 2021 *notes, videos and* Leitfaden *of [ExpHoriz1; ExpHoriz2]; for a broader introduction and additional references, we refer to the Program and list of references of [Prom20].*

References & Notes

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[A] "On peut considérer que la géométrie nouvelle est avant toute autre chose, une synthèse entre ces deux mondes, jusque là mitoyens et étroitement solidaires, mais pourtant séparés : le monde "arithmétique", dans lequel vivent les (soi-disants) "espaces" sans principe de continuité, et le monde de la grandeur continue, ou vivent les "espaces" au sens propre du terme, accessibles aux moyens de l'analyste [...]. Dans la vision nouvelle, ces deux mondes jadis séparés, n'en forment plus qu'un seul [...,] vision d'une "géométrie arithmétique" (comme je propose d'appeler cette géométrie nouvelle)."

[B] "Les plus profonds (à mes yeux) parmi ces douze thèmes, sont celui des motifs, et celui étroitement lié de géométrie algébrique anabélienne et du yoga de Galois-Teichmüller."

[C] " Une telle supposition avait l'air à tel point dingue que j'étais presque gêné de la soumettre aux compétences en la matière. Deligne consulté trouvait la supposition dingue en effet, mais sans avoir un contre-exemple dans ses manches. Moins d'un an après, au Congrès International de Helsinki, le mathématicien soviétique Bielyi annonce justement ce résultat, avec une démonstration d'une simplicité déconcertante tenant en deux petites pages d'une lettre de Deligne – jamais sans doute un résultat profond et déroutant ne fut démontré en si peu de lignes !"

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